Hansjoachim Walther

Ten Applications of Graph Theory

Hansjoachim Walther

itute of Technology, Ilmenau, G.D.R.

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Editor's Preface

Growing specialization and diversification have brought a host of monographs and textbooks on increasingly specialized topics. However, the "tree" of knowledge of mathematics and related fields does not grow only by putting forth new branches. It also happens, quite often in fact, that branches which were thought to be completely disparate are suddenly seen to be related.

Further, the kind and level of sophistication of mathematics applied in various sciences has changed drastically in recent years: measure theory is used (non-trivially) in regional and theoretical economics; algebraic geometry interacts with physics; the Minkowsky lemma, coding theory and the structure of water meet one another in packing and covering theory; quantum fields, crystal defects and mathematical programming profit from homotopy theory; Lie algebras are relevant to filtering; and prediction and electrical engineering can use Stein spaces. And in addition to this there are such new emerging subdisciplines as "completely integrable systems", "chaos, synergetics and large-scale order", which are almost impossible to fit into the existing classification schemes. They draw upon widely different sections of mathematics.

This program, Mathematics and Its Applications, is devoted to such (new) interrelations as exempla gratia:

– a central concept which plays an important role in several different mathematical and/or scientific specialized areas;
– new applications of the results and ideas from one area of scientific endeavor into another;
– influences which the results, problems and concepts of one field of enquiry have and have had on the development of another.

The Mathematics and Its Applications programme tries to make available a careful selection of books which fit the philosophy outlined above. With such books, which are stimulating rather than definitive, intriguing rather than encyclopaedic, we hope to contribute something towards better communication among the practitioners in diversified fields.

Because of the wealth of scholarly research being undertaken in the Soviet

Union, Eastern Europe, and Japan, it was decided to devote special attention to the work emanating from these particular regions.

Thus it was decided to start three regional series under the umbrella of the main MIA programme.

Graph theory, the topic of the present volume in the MIA (East European Series), is a fully-established subdiscipline in itself and related to many parts of (pure) mathematics. It is also eminently and directly applicable in a large number of concrete situations. This is what this book is about. For people who want to know how to apply graph theory (or who need examples for lectures on the topic) it has an almost ideal structure in that it first sets out the original problem, then proceeds to discuss the graph theory involved, and finally presents the algorithm(s) which exist to solve them.

The unreasonable effectiveness of mathematics in science ...

Eugene Wigner

Well, if you knows of a better 'ole, go to it.

Bruce Bairnsfather

What is now proved was once only imagined.

William Blake

As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited.

But when these sciences joined company they drew from each other fresh vitality and thenceforward marched on at a rapid pace towards perfection.

Joseph Louis Lagrange

Amsterdam, March 1983

Michiel Hazewinkel

Preface

The present book is a translation of the textbook *Anwendungen der Graphentheorie* which was published by Deutscher Verlag der Wissenschaften in 1978. It is meant for students studying all branches of operations research, and for graduates and practical men to give them a means for modelling and solving organization and optimization problems, in particular with a combinatorial component.

The application of graph theory implies two aspects. On the one hand, it is applied graph theory with attention being given to the numerical ascertainment of the characteristic values of a given graph (e.g. the question arises of how to find a minimal set of arcs in a graph after the removal of which the graph is circuit-free; cf. Chapter 9). On the other hand, it implies the application of theorems and algorithms of graph theory in other scientific domains (when determining an optimal sequence of computation in algorithm, a decisive role is played, for example, by loops, and the question arises how many feedback arcs have to be cut to make the running of the algorithm loop-free; cf. Chapter 9). Both aspects are connected with each other and are discussed in the book.

The short introduction contains the most necessary concepts of graph theory which will be used in the text, and those concepts, which are required for one chapter only, are defined. Chapter 1 forms the basis for all other chapters dealing with flow problems, while the remaining chapters are, in essence, independent.

On the basis of well-known theorems of graph and network theory which will be discussed in more detail in the first chapter, a number of theorems will be formulated and proved, resulting in practicable algorithms. These will then be given at great length and elucidated by means of examples. Particular attention has been attached to the choice of examples used. They are by no means simple, but have been chosen, where possible, such that all difficulties, and also all subtleties of the algorithm, become apparent. Although the algorithms given are constructed such that they can be directly executed by computer engineering means, we had to desist from further preparation for those with practical experience, since the character of a textbook should be preserved.

In some chapters (Chapter 5 and 10) the bounds of current applications research are touched upon but, in general, mention is made in the bibliography of more or less dispersed methods which give an insight into the variety of spheres of application. Of course, completeness has not been aimed at and numerous examples of application (e.g. graph spectra in chemistry) could not be included. Much attention has been given to flow and tension and supply and transportation problems, and to planarity studies.

All theorems mentioned are proved and the reader who is primarily interested in the approaches and algorithms may skip the proofs for the time being. The more mathematically inclined reader, however, has the chance of checking his own capabilities in attempting to solve the numerous exercices contained in the text. The large number of figures are not only helpful for understanding, but they also point out the advantages offered by the possibility of representing graph and network problems in an illustrative way. The reader should thus become acquainted with standard problems and procedures and should also be enabled to recognize the combinatorial core of many other problems and, finally, to solve them on the basis of graph- and network-theoretical approaches.

In the framework of the translation the problematic nature of the complexity of problems (NPC problems), which has come to the fore at an even higher degree in recent years, has been indicated only briefly (cf. Chapter 6 "Assignment and travelling salesman problems").

Many thanks to all my colleagues who helped me to elaborate on the German original version and I wish to thank Mrs. Brigitte Schönefeld for having conscientiously typed the manuscript. My special thanks are due to Mrs. Ursula Nixdorf, who made the translation, and to my wife Ute who showed a lot of understanding for this work.

Ilmenau, February 1983 **Hansjoachim Walther**

Contents

Chapter 0

Introduction

Many problems encountered in various spheres of life involve graphs, directly or indirectly. Who would not have tried, as a schoolboy, to draw the "House of Santa Claus" with a single stroke (cf. Fig. 0.1) or to solve the problem of the three houses and the three factories (cf. Fig. 0.2) requiring each house to be connected with each factory by strokes which do not cross? Though less obviously, some of them may be formulated as problems of graph theory like, for example, the following:

A group of seven chess-players wants to find the best lightning-chess-player among them. Each one of them has to play twice for five minutes against each of the others. How can the match be organized so that after seven rounds each of the players has played one game against each of the others?

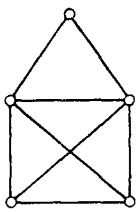

Fig. 0.1

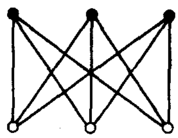

Fig. 0.2

By assigning a suitable graph to the problem, we transform the task into the following: assign a node to each player, connect each node with each of the others by an edge and then try to split up the resulting graph, which is a complete graph with seven nodes, in such a way that in each part there are exactly three edges and six nodes. In the language of graph theory: try to decompose the graph into seven matchings.

To the credit of the chess-players, it should be said that in fact they solved this problem long ago, for any number of players. In the case of an odd number of players, each of them plays exactly as many times as black, and in the case of an even number of players, half of them have white once more often than they have black and the other half have black once more often than they have white.

Let us now define some terms which we will use more frequently:

By a graph $G = G(\mathfrak{X}, \mathfrak{U})$, we mean a set $\mathfrak{U}$ of *edges* (or *arcs*), a set $\mathfrak{X}$ of *vertices* and an *incidence function* f assigning to each of the edges $u \in \mathfrak{U}$ an ordered or an unordered pair (X, Y) of vertices X and Y from $\mathfrak{X}$. X and Y are called the *end-points* of the edge u; if $X = Y$, u is called a *loop*. If only ordered pairs of vertices are assigned to them, the graph is called *directed* or *oriented*, otherwise *undirected*. In the case of a directed graph G and $f(u) = (X, Y)$, X is called the *initial vertex* and Y the *terminal vertex* of the arc u.

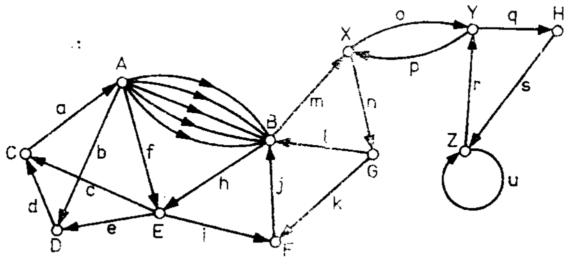

Fig. 0.3

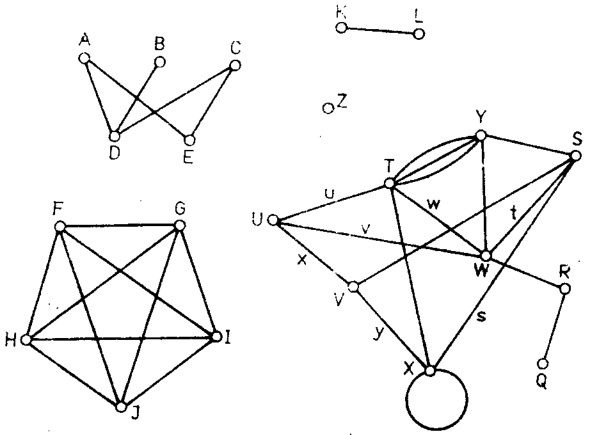

Fig. 0.4

Directed graphs will be represented as in **Fig. 0.3**, the undirected ones as in Fig. 0.4. Graphs which are partially directed and partially undirected will not be considered here. The elements of $\mathfrak{X}$ and $\mathfrak{U}$ will often be chosen from the set of natural numbers as shown in Fig. 0.5.

If several edges of an undirected graph have the same end-points, for example, the three edges of Fig. 0.4 having the end-points T and Y, the set of those edges is called a *multiple edge*. (In the case of the example cited, a *triple edge*.) Correspondingly, arcs which have the same initial and terminal vertices are called *multiple arcs*. Thus, in Fig. 0.3, the arcs with the initial point A and the terminal point B

form a *quintuple arc*, the two arcs with the terminal points X and Y, respectively, do not form a double arc, because the arcs are distinctly oriented.

Let $u_1, u_2, \ldots, u_r$ be edges of an undirected graph G, with for each subscript i $(i = 2, 3, \ldots, r - 1)$ the edge u_{i-1} having one of its end-points in common with u_{i-1} and the other in common with u_{i+1}. Then we call $W = (u_1, u_2, \ldots, u_r)$ a ***chain*** of G. Thus, u, v, w, u, x, y in Fig. 0.4 form a chain. A chain is called a ***simple chain*** when it does not use the same edge twice, for example (w, u, v, t). A simple chain that does not encounter the same vertex twice is called *elementary*, for example (t, w, u, x). If a simple chain is closed, without the same vertex being used twice, we call it a *circuit*, as for example in Fig. 0.4 (t, w, u, x, y, s).

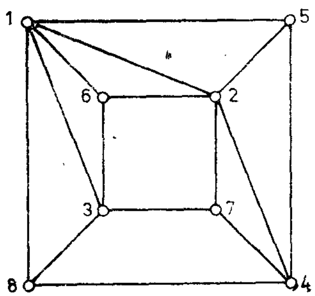

Fig. 0.5

A graph G is called *connected* if there exists an elementary chain for each pair of distinct vertices of G. The graphs in Figs. 0.1, 0.2 and 0.5 are connected; that in Fig. 0.4 is not connected, it possesses five components. A *component* is a maximal connected subgraph of a graph.

Two vertices X, Y are called *neighbouring* or *adjacent* if there is an edge u with $f(u) = (X, Y)$; X and Y are said to be ***incident with*** u. A vertex that is not incident with any edge is called *isolated*, as it is, for example, vertex Z in Fig. 0.4. The number of edges incident with a vertex X is called *degree* $v(X)$ of X. In Fig. 0.4, $v(S) = 4$ holds. Furthermore, we put $v(X) = 5$ (the incidence with a loop shall yield an increase of the degree of 2).

Let us consider some further concepts relating to directed graphs (details will be given in the first chapter):

A sequence $u_1, u_2, \ldots, u_r$ of arcs forms an *elementary chain* (or simply, *chain*) if it goes over into a path of the resulting undirected graph, with the orientations of the arcs being ignored. If all arcs are oriented in the direction of the traversing of a chain, then the chain is a ***simple path*** (or an elementary chain). In Fig. 0.3, a, d, e, h, m form a chain, but not a simple path, connecting the vertices A and X, and p, n, l, h, c form a simple path connecting the vertices Y and C.

A directed graph is called ***connected*** if there exists, for each pair of vertices, a chain joining these vertices, if the undirected graph resulting from ignoring all orientations is connected. We call it *strongly connected* if for each pair of vertices there exists a simple path joining these two points.

The graph represented in Fig. 0.3 is strongly connected, that of Fig. 0.6 only connected.

Explaining the vertices as states of a Markov's chain and joining X and Y by an arc — if it should be possible to go from state X over to state Y with positive probability — would mean that the resulting graph would be strongly connected so that the states of the Markov's chain form a class of essential states.

We shall deal exclusively with finite graphs. These are graphs with a finite number of vertices, edges or arcs.

Finally, let us explain two more concepts: We shall frequently use the terms *maximal* and *maximum* (correspondingly *minimal* and *minimum*). A clear distinction has to be made between them. We always use maximal in the sense of relatively maximal, whereas maximum is used in the sense of absolutely maximal. We explain this by means of an example which is typical for the following expositions: We consider the graph G in Fig. 0.6. The vertices X, Y, Z span a maximal strongly connected subgraph G' since there is no "larger" strongly connected subgraph of G containing G'. But because of the strong connectivity, the graph G' is not a maximum subgraph, since the subgraph spanned by the vertices A, B, C, D, E is also strongly connected and contains five vertices.

We take another example (cf. Fig. 0.5): We search for those sets of vertices which *represent all circuits*, i.e., for a set $\mathfrak{Y}$ of vertices containing at least one vertex from each circuit.

Evidently, {5, 6, 7, 8} represents all circuits. This set is even minimal, because there is no proper subset representing *all* circuits. On the other hand, {2, 3, 5} or {2, 3, 4} forms a minimum set representing all circuits.

If we think, however, of a *vertex valuation* $w(X)$ (*weighting*) to be given, say in the form $w(X) = v(X)$, i.e. if its degree is assigned as valuation to each vertex, then for the above minimal sets such valuations result that {5, 6, 7, 8} with a total weight of 12 becomes a minimum set, just like {2, 3, 5}.

In the course of our expositions, we shall constantly come across such *weightings* or *valuations* of vertices (say by *potentials*) or of edges or arcs (by *flows*, *tensions*, *lengths*, *capacities*, *costs*, etc.), and it is through the valuation of the elements of a graph that we leave the *hard graph theory* as denoted by G. A. Dirac and come to the *network theory* or to the *application of graph theory*.

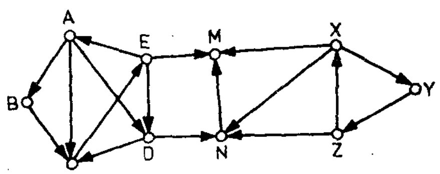

Fig. 0.6

Chapter 1

Flows and tensions on networks

1.1. BASIC CONCEPTS

For setting up a theory of flows and tensions the concepts of cycle and cocycle are of fundamental significance. Therefore, we want to put them at the top of our considerations.

Let $G(\mathfrak{X}, \mathfrak{U})$ be a directed graph with m arcs $u_1, \ldots, u_m$. A *cycle* μ is a cyclically ordered set of arcs $u_{i_1}, \ldots, u_{i_k}$ of G that are pairwise different, with the property that the arc u_{i_j} has one of its end-points in common with one of the end-points of $u_{i_{j-1}}$, and the other of its end-points in common with one of the end-points of $u_{i_{j+1}}$. Here, j is to be reduced modulo k (Fig. 1.1).

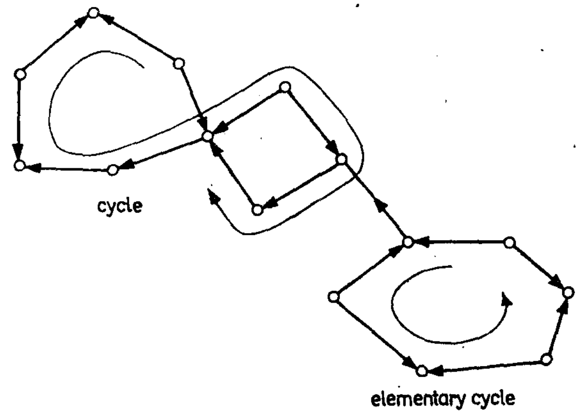

Fig. 1.1

We denote by μ^* the set of all arcs in μ, without taking into consideration the order:

$$\mu^* = \{u_{i_1}, \ldots, u_{i_k}\}.$$

By the cyclic order of the arcs the cycle μ is given a direction of traversing decomposing μ^* into two classes μ^+ and μ^-. The set μ^+ contains all arcs in μ^* that are in the direction in which the cycle μ is traversed, and the set μ^- contains all arcs in μ^* that are in the opposite direction in which μ is traversed. Thus, we get the relation

$$\mu^* = \mu^+ \cup \mu^- .$$

Let $\boldsymbol{\mu}^+$ and $\boldsymbol{\mu}^-$ be the characteristic vectors of μ^+ and μ^-, respectively. We assign to the cycle μ the vector

$$\boldsymbol{\mu} = \boldsymbol{\mu}^+ - \boldsymbol{\mu}^-$$

such that if we put

$$\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_m),$$

it holds:

$$\mu_i = \begin{cases} 1 & \text{if } u_i \text{ is contained in cycle } \mu \text{ and is in the direction in which this cycle is traversed,} \\ -1 & \text{if } u_i \text{ is contained in cycle } \mu \text{ and is in the opposite direction in which this cycle is traversed,} \\ 0 & \text{if } u_i \text{ is not contained in cycle } \mu. \end{cases}$$

If the direction of traversing is inverted, a cycle μ will be transformed into a cycle $\tilde{\mu}$. Evidently, it holds for the related vectors:

$$\tilde{\boldsymbol{\mu}}^+ = \boldsymbol{\mu}^-, \qquad \tilde{\boldsymbol{\mu}}^- = \boldsymbol{\mu}^+, \qquad \tilde{\mu} = -\boldsymbol{\mu}. \tag{1}$$

We can also write

$$\tilde{\mu} = -\mu .$$

The set of the terminal vertices of the arcs of a cycle μ is called the *set of vertices of* μ.

A cycle μ is called an *elementary cycle* if, when traversing it, no vertex is encountered more than once. A cycle μ is called *minimal* if the set μ^* of its arcs contains no proper subset the arcs of which may be arranged themselves to form a cycle The following theorem is evident (task!):

THEOREM 1.1. *A cycle is minimal if and only if it is an elementary cycle.*

If for a given cycle μ pairwise arc-disjoint cycles $\mu^1, \ldots, \mu^q$ with $q \geqq 1$ can be found such that the appertaining vectors satisfy the equation

$$\boldsymbol{\mu} = \boldsymbol{\mu}^1 + \boldsymbol{\mu}^2 + \cdots + \boldsymbol{\mu}^q,$$

it is said that μ can be partitioned into the cycles $\mu^1, \mu^2, \ldots, \mu^q$.

Furthermore, it holds:

THEOREM 1.2. *Each cycle μ can be partitioned into elementary cycles.*

Proof. Follow a traversing of μ and split off an elementary cycle μ^1 when reaching for the first time a vertex already traversed. If μ is not identical to μ^1, then pursue the traversing and split off an elementary cycle μ^2 correspondingly, etc., until all arcs of μ are traversed (Fig. 1.2).

REMARK. Fig. 1.2 shows that, in general, the partition into elementary cycles is not unique.

A cycle is called a *conformally directed cycle* μ (*c-cycle* for short) if all of its arcs are directed in the sense of the orientation of μ (cf. Fig. 1.3). Thus, it holds $\mu_i \neq -1$ for all i's. We call a conformally directed elementary cycle an *elementary circuit* (shortly also, *circuit* (cf. Fig. 1.3)).

In order to get the concept of a cocycle, we proceed as follows. Let the set $\mathfrak{X}$ of the vertices of $G(\mathfrak{X}, \mathfrak{U})$ be partitioned into two non-empty classes $\mathfrak{A}$, $\mathfrak{B}$ such that the following relations hold:

$$\mathfrak{A} \cup \mathfrak{B} = \mathfrak{X}, \qquad \mathfrak{A} \cap \mathfrak{B} = \emptyset, \qquad \mathfrak{A} \neq \emptyset, \qquad \mathfrak{B} \neq \emptyset.$$

Furthermore, let ω^* be the set of those arcs that have one of their end-points in $\mathfrak{A}$ and the other end-point in $\mathfrak{B}$ (we assume that $\omega^* \neq \emptyset$). Considering the classes

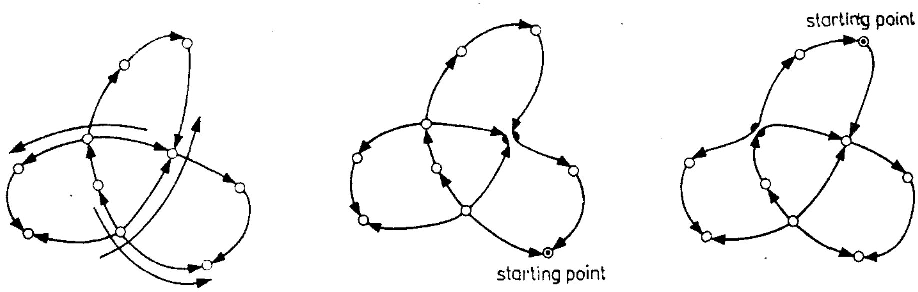

Fig. 1.2

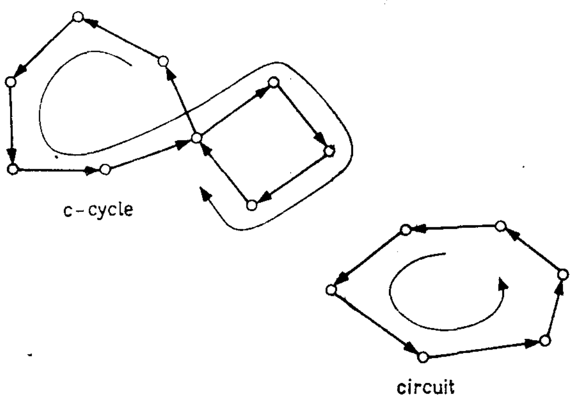

Fig. 1.3