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The Logic of Quantum Mechanics

Enrico G. Beltrametti and Gianni Cassinelli

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The Logic of Quantum Mechanics

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Editor's Statement

A large body of mathematics consists of facts that can be presented and described much like any other natural phenomenon. These facts, at times explicitly brought out as theorems, at other times concealed within a proof, make up most of the applications of mathematics, and are the most likely to survive changes of style and of interest.

This ENCYCLOPEDIA will attempt to present the factual body of all mathematics. Clarity of exposition, accessibility to the non-specialist, and a thorough bibliography are required of each author. Volumes will appear in no particular order, but will be organized into sections, each one comprising a recognizable branch of present-day mathematics. Numbers of volumes and sections will be reconsidered as times and needs change.

It is hoped that this enterprise will make mathematics more widely used where it is needed, and more accessible in fields in which it can be applied but where it has not yet penetrated because of insufficient information.

The present volume is meant to be the first of a series on quantum mechanics and its applications. It deals with the foundations as well as the fascinating logic of quantum mechanics.

This volume is more general and at places less factual than other volumes of the ENCYCLOPEDIA; nevertheless the thorough presentation will guide the reader to gain an overview of a theory that, together with relativity, is regarded as the greatest achievement in physics in this century.

GIAN-CARLO ROTA

Foreword

For many years the physical interpretation of quantum theory has been dominated by the "wave-particle duality" attitude of the "Copenhagen school." This point of view is eloquently described in Bohr's collection of essays on the subject.[1] Despite persistent concerns with apparent paradoxes and limitations to this interpretation (as exemplified by Schrödinger's cat, the paradox of Einstein, Podolsky and Rosen, among others), the Copenhagen view persists *de facto* in the daily life of the modern physicist. Traditional quantum theory has so successfully explained such a vast amount of data in atomic and molecular physics, solid state physics (and to a lesser extent, elementary particle physics) that little doubt can exist concerning its essential validity.

As a consequence of the overwhelming practical success of the "Copenhagen interpretation," the latter has acquired the status of dogma. For many years, therefore, most physicists have found it expedient to relegate the puzzling aspects of the theory to philosophers, and mathematicians. Nevertheless, a persistent interest in this subject has produced a significant and fascinating literature, reviewed by Jammer.[2] In recent years, concerns over the proper meaning to be ascribed to quantum theory have produced an increasingly deep and incisive series of investigations.

These investigations fall generally into two categories: either (1) discussion of the philosophical content of the theory, or (2) analyses of the mathematical variants of the theory and their connection with differing interpretational schemes. Physicists tend to be detached from a commitment to philosophical issues, because of their realization of the transient character of the meanings attached to theories of the day. As a simple illustration of this we can mention the profound differences between nonrelativistic and relativistic quantum mechanics. The former is the subject of most of the analysis alluded to in the foregoing. As pointed out by the authors of this treatise, profoundly different conceptual problems appear when relativity is properly taken into account. These differences are largely associated with particle creation and destruction processes and the retarded character of the sequential interactions mediated by virtual particles such as photons. Although such features are correctly accounted for by a suitable extension of quantum mechanics, the associated technical changes have a profound impact on the usual interpretative analyses. As an example we can mention the role of the position coordinate in the two theories. In nonrelativistic quantum mechanics the position variable is a full-blooded dynamical variable, while in quantum field theory it is a mere parameter (along with the time coordinate) labeling points in the space time continuum for the true dynamical variable, the quantized field itself. Associated with this situation is the conceptually distinct role played in nonrelativistic quantum mechanics by the position-momentum and energy-time uncertainty relations. Such circumstances indicate that the nonrelativistic theory is structurally incomplete and that undue focus on its detailed mathematical structure may not contribute to the mainstream of scientific progress.

Despite these reservations, the axiomatization of nonrelativistic quantum mechanics has had a decisive and creative role in the development of modern theoretical physics. Indeed the pioneering works in this field by von Neumann[3] and Dirac[4] were of such brilliance as to dominate the subject for decades, perhaps precluding heretical views. In von Neumann's book we find an almost definitive interpretation of quantum mechanics in terms of abstract relations in a Hilbert space. In addition, the essence of measurement theory is laid out in such a modern form that it is still prey to scholarly commentaries, improvement, and dispute. Dirac's incisive essay still provides the essence of the modern theorist's repertoire. Unmatched in the preeminence of physical intuition over mathematical rigor, through the stimulation of distribution theory, it has allowed modern physicists to live in a space other than Hilbert space. Indeed, nowadays one resorts to Hilbert space only to resolve mathematical niceties not clearly visible in the freewheeling world of unnormalized wave functions. This example provides foundation to the hope that future generalizations of the mathematical foundations of quantum mechanics will provide new insights, both to the conceptual and practical, by the continuing evolution of quantum field theory discoveries in elementary particle physics. In conjunction with these developments it must be mentioned that the detailed and beautiful texture of quantum mechanics can only be appreciated in the context of the group-theoretical structures dictated by both relativity theory and the "internal" symmetries discovered in weak, electromagnetic and strong interactions. These and other developments are mainly due to Wigner in a monumental sequence of works spanning the last fifty years.

At the moment of writing, quantum field theory seems to be heading in novel directions. The search for unification of the dynamics of all interactions has proceeded through promising directions through the route of non-Abelian gauge theories. The most convenient technical formulation of such theories is through the path integral formulation of field theory. Massive generalizations and possibly also modifications of our quantum-mechanical inheritance seem imminent.

In view of the illustrious history of quantum physics and the promises of contemporary problems, it is especially useful to have at hand the clear and thorough treatment of the structure of quantum theory presented in the treatise of Beltrametti and Cassinelli. Past experience shows that the refinement and analysis of the basic structural precepts of the foundations of physics can have a decisive impact on the evolution of science. Often these analyses require some time to have an impact, as exemplified by Mach's ideas on mechanics and especially his concept of the origin of inertial frames, now firmly embedded in the cosmology of modern relativity theories.

The present work is laid out in clear lines, avoiding the abyss of philosophical interpretations. The exposition adheres to a clean exposition of the mathematical content of serious formulations of rational physical alternatives of quantum theory as elaborated in recent researches, to which the authors have made noteworthy contributions. This work builds on earlier influential essays of Mackey[5], Jauch[6], Piron[7], Varadarajan[8], and others.

The treatment of the subject falls into three distinct, logical parts. In the first part, the modern version of accumulated wisdom is presented, avoiding as far as possible the traditional language of classical physics for its interpretational character. In the second part, the individual structural elements critical for the logical content of the theory are laid out, with special attention to the empirical evidence underlying each ingredient of the theory. Finally, in part three, the results of section two are used to reconstruct the usual Hilbert space formulation of quantum mechanics in a novel way. The general community of physicists and mathematicians concerned with this most decisive scientific issue will be grateful to the authors for their thoughtful and incisive analysis of this fascinating and profound problem.

PETER A. CARRUTHERS
General Editor, Section on Mathematics of Physics

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PREFACE

"The Logic of Quantum Mechanics" appeared as the title of a scientific work in 1936, with the paper of Garrett Birkhoff and John von Neumann (9).* The use of the same title for this volume outlines the fact that a great part of the subject matter we shall deal with pertains to a research field that originated in that historic paper. This title, however, is not to be interpreted as focusing on the propositional calculus that mirrors the structure of quantum mechanics, the so-called "quantum logic" with the word "logic" used in technical sense; rather, the title should be interpreted as focusing on the mathematical foundations of quantum mechanics. The complex edifice of this theory contains simpler substructures that have direct physical bases; each of them can explain some aspects of the behavior of quantum systems. This was also the idea of the classical books by G. W. Mackey (3), J. M. Jauch (3), and V. S. Varadarajan (3), which appeared in the sixties. The present volume includes results of more than a decade of active research which followed these classical works.

The volume is divided into three parts. The first contains an exposition of the basic formalism of quantum mechanics using the theory of Hilbert spaces and of linear operators in these spaces. We shall not follow the old tradition of striving to use concepts of classical mechanics to explain quantum facts (as in the so-called principle of correspondence or the wave-particle dualism)—a tradition that reflects the unusually long delay suffered by quantum mechanics before it acquired autonomy and internal coherence, breaking with the language of the theory to be superseded. The second part follows the program of decomposing quantum theory into its conceptual constituents, singling out the basic mathematical structures, isolating what may be founded on direct empirical evidence, and controlling how single assumptions contribute to shape the theory. In the third part we face the problem of recovering the Hilbert-space formulation of quantum mechanics, starting from the simpler and more general theoretical schemes examined in the second part.

Almost every chapter contains exercises with hints. We consider them an integral part of the text. In some of them we deal with proofs of mathematical facts that are only stated in the text; in the others we provide examples that can help the comprehension of the matter.

*Papers and books mentioned in this Preface appear also in following chapters. Here we give in parentheses the number of the chapter in which the reference can be found.

This volume is primarily written for theoretical physicists and students of quantum mechanics who want to have a critical and deeper understanding of quantum theory. The knowledge of quantum mechanics is however not assumed as a prerequisite. We think that the volume can be of interest also to mathematicians. For instance, a functional analyst can find in Part I one of the most notable applications of the theory of linear operators in Hilbert space, while people dealing with lattice theory, or related fields, can find in Part II pertinent physical applications.

Throughout this volume the notion of physical system will be used. It designates the physical object to be studied, the actual object of the theory. A photon, an electron, a proton, a nucleus, a molecule, and an aggregate of molecules appear to be, in most circumstances, clear examples of physical systems. Though the notion of physical system is familiar, it contains some idealization. It is generally understood that a physical system is a portion of the physical universe, whose interaction with the rest of the universe, and with the observer in particular, can be neglected or, in any case, is such as to cause no trouble about the identity of the separated portion. The notion of physical system thus inherits, in particular, all difficulties related to the separation between the observer and what is observed. Nevertheless it works without relevant ambiguities in the cases that form the domain of elementary quantum mechanics, where destruction and creation processes are absent, or at least are not of primary importance, so that the physical system remains fixed in time. Since the passage to higher and higher energies makes destruction and creation processes more and more relevant, the expression "elementary" could here be replaced by "nonrelativistic." And, indeed, the concept of physical system becomes blurred in situations involving high-energy elementary particles, where starting with a given set of particles one can end with another set of quite different ones. Thus we shall have to do with elementary, nonrelativistic quantum mechanics.

There are some basic ingredients in the description of physical systems: we mention the concepts of "state," of "physical quantity," and of "proposition," noting that only for the first is the choice of the name universal.

By "state" is meant the result of the set of experimental procedures used to isolate and prepare the physical system. Notice that, in order to have a well-defined notion, these preparation procedures should fulfill the strong requirement of eliminating any memory of what happened to the system before the preparation procedure was started.* The notion of state collects all those attributes of the physical system that are accidental, in the sense

*A requirement of this sort is nonobvious in the absence of space-time confinement, as may happen in relativistic quantum field theories.

that they may be different in different situations and may change with time; the permanent attributes are instead embodied in, and form the body of, the very notion of physical system. It should be clear that the point where the definition of system ceases and the definition of state begins is, to a large extent, a matter of convention, and may depend on the historical development of scientific knowledge (as a familiar example, think of the proton and the neutron, which appeared on the scientific scene as different physical systems, with electric charge regarded as an essential attribute, but are usefully viewed as two states of one and the same system, the nucleon, when strong interactions are dominant).

By "physical quantity" is meant any quantity that can be measured: energy, position, momentum, angular momentum are familiar examples. Equally common synonyms of "physical quantity" are "observable" and "dynamical variable."

There are certain physical quantities, those that admit just two outcomes, that are of special importance and merit a name of their own. We call them "propositions." There are several alternatives to this term, each one alluding to a particular way of picturing the same idea: we may mention the terms "property," "event," "test," "question," "filter," and "yes-no experiment," all used in the literature. Characteristic of the notion here considered is a dichotomy: a proposition is either true or false, a property is either possessed or not, an event either occurs or not, a test or a filter is either passed or not, etc.

Let us anticipate that in the usual Hilbert-space formulation of quantum mechanics states are density operators, physical quantities are self-adjoint operators, and propositions are projection operators.

In some schemes of description of physical systems another basic ingredient is introduced. It may be called an "operation," though an array of somewhat different interpretations can be found: "operation" might stand for the set of instructions for performing an experiment on the physical system, or, more specifically, it might stand for the procedure of measuring a physical quantity, or it might even stand for the transformation of the state of the physical system caused by the measurement process.

The various schemes of description of physical systems that we examine in Part II make different choices about which ingredients, among states, propositions, physical quantities, and operations are taken as primitive undefined notions, and which ones are taken as derived. These approaches emphasize different mathematical structures; it will be seen in particular that what is typical of propositions is an ordered structure, which will be our main guiding theme. Relative to this ordered structure, the passage from classical to quantum systems corresponds to the abandonment of the distributive law, as already pointed out in Birkhoff and von Neumann's paper.

We favor, in this volume, the probabilistic standpoint according to which states are probability measures on the set of propositions. The nonprobabilistic standpoint, developed by C. Piron, has been eloquently exposed in his recent book (12). We only touch (in Chapter 19) on the operational approach due to G. Ludwig and B. Mielnik. We leave aside the developments of this approach due to E. B. Davies, C. M. Edwards, J. Lewis, and others (19). We also leave aside the operational statistics worked out by D. Foulis, C. Randall, and their school, which constitutes a general language promising an alternative approach to quantum theory. We hope that, within these limits, this volume contributes to the understanding of the logic of quantum mechanics.

It is a pleasure to thank Professor G. C. Rota for asking us to write this volume, and many people who helped us in various ways: R. Greechie for numerous general suggestions and for a critical analysis of the manuscript; G. W. Mackey, with whom we discussed the organization of the volume when it was in an early stage of development; M. Maczynski for many suggestions on the structure of the volume; S. S. Holland for valuable correspondence; B. Mielnik for a number of enlightening discussions; S. Bugajski for his comments and for criticisms and suggestions on Sections 3.3 and 3.4 and Chapter 18; G. T. Ruttimann for his decisive help with Chapter 19; P. Lahti for useful discussions, especially on the subjects of Sections 3.4, 3.5 and 14.7; P. Truini for his help on Section 22.3; and D. Aerts for his help on Chapter 24.

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ENRICO G. BELTRAMETTI

GIANNI CASSINELLI

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