SEARCH THEORY

Some Recent Developments

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Edited by

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Preface

Major efforts of the mathematical community in the United States in defense research during World War II led to the development of many new areas of applied mathematics, and provided sufficient justification for future funding of research in these areas. Several areas of pure and applied research became known under the general title of *operations research*. Search theory emerged as a major part of operations research in the work of the Antisubmarine Warfare Operations Research Group (ASWORG), directed by P. M. Morse, under Admiral E. King, Chief of Naval Operations and Commander in Chief, U.S. Fleet. Specific tasks in the development of procedures for antisubmarine search were undertaken by B. O. Koopman, J. M. Dobbie, and others. The original reports of the Operations Research Group are available as Refs. 1 and 2. Koopman laid down the foundations of search theory in Ref. 3. An exposition of the human side of this effort can be found in Morse's reminiscences [4,5].

Since the declassification in 1958 of the original search theory contributions during and after World War II, the field developed rapidly into an independent discipline closely connected with various problems of optimal control, game theory, differential games, and statistics. On the practical side, there were many well-publicized achievements of search plans that followed search theory recommendations. We refer to the opening review of H. R. Richardson in this volume for the literature on these successes, in some of which the author of the review took an active part. There is a considerable body of literature on the traditional approach to search theory initiated by Koopman. We cite particularly the classical book of Stone [6]. Other recent monographs on search theory are those of Ahlswede and Wagner [7], Washburn [8], and Mangel [9]. The classic contribution of Koopman [2] was published in expanded and updated form in Ref. 10. Different approaches to search theory, particularly from the game-theoretic point of view, were pioneered by R. Issaacs. Important progress in this direction is due to Gal [11]. As references to differential games and pursuit games, some of which are considered in contributions to this volume, we recommend the books of Issaacs [12] and Hajek [13].

Search theory as it stands now has a few definitive principles and criteria that evolved since Koopman's time. Attempts to incorporate dynamic programming and game-theoretic considerations have introduced many mathematical techniques in this area. It is still open to further expansion and refinement of definitions, algorithms, and concepts. A variety of techniques borrowed from such diverse areas as classical and quantum dynamical systems, ergodic theory, number theory, and statistical decision theory have been introduced recently. This volume summarizes the latest developments in search theory, including the classical, differential equations, optimal control, game-theoretic, and statistical and ergodic theory approaches.

Chapters in this volume are organized as follows. The contributions are preceded by the general introductory article by H. R. Richardson, originally prepared for the *Encyclopedia of Statistical Sciences* (Wiley). It contains a review of the contemporary literature and clearly describes the basic solution to an optimal search problem with an exponential detection function. Also, references to nonmaritime applications of search theory (to biology, mineral exploration, and maintenance/inspection) are given. The contribution of L. P. Stone covers one-sided and two-sided detection problems, that is, whether the target is evading or not. This review article describes continuous and discrete time strategies as well as surveillance, and gives an exposition of generalized search optimization (GSO) techniques developed by the author. The two-sided search strategies are examined in the chapter by S. Gal. He solves a variety of "hide and seek" games in many discrete and continuous bounded and unbounded domains. His work (cf. [11]) has its origins in the classic model game "The Princess and the Monster," introduced by Issaacs [12]. An interesting feature appearing in Gal's algorithm is a variety of randomized strategies, useful in many other search plans.

A different approach to optimal search strategies is developed by M. Mangel. His use of differential equations, which builds on the work of O. Hellman and J. Keller, provides a consistent framework for solving complex search problems in a unified way. The ability to use ray and WKB methods allows researchers to apply a large arsenal of modern numerical methods to practical problems. M. Mangel kindly permitted us to publish this review, intended originally to be a book. This chapter also contains references to applications of search theory to economics and other social sciences. The chapter by D. V. Chudnovsky and G. V. Chudnovsky describes systematic ways to generate tours (search paths or ϵ-Peano curves) for optimal search in bounded domains. A variety of number-theoretic problems connected with billiard strategies and uniform distributions is discussed. A novel class of random search plans is described in the contribution of S. Lalley and H. Robbins. They present a particularly elegant and easy-to-implement randomized strategy to generate nearly optimal tours, and prove new ergodic theorems. Although their chapter treats only circular domains, we are happy to report that very recently they generalized their results to arbitrary bounded domains.* These and similar strategies performed very impressively in a variety of numerical simulations.

This book should serve as an invitation to mathematicians and statisticians to look into the depths of search theory; you are bound to find something new and, perhaps, even something useful. Experts in thi area will steer you. Their contributions constitute the bulk of this volume and we cordially thank them: S. Gal, M. Mangel, H. R. Richardso and L. Stone. We are fortunate to include their contributions in this volume. The books by Ruckle [14], Haley and Stone [15], and Hellman [16] must also be added to the recommended list.

The editors of this volume became interested in a problem of optimal searching when the Columbia University Applied Mathematics Group started a project in this direction. We became a part of the group that pursued these and other applied mathematics problems; it included M. Friedman, S. Lalley, K. Prendergast, and H. Robbins. The overall efforts of the group, including those in search theory, were coordinated by W. Brown of Hudson Institute, a cosponsor of the project. The work of the Columbia Applied Mathematics Group in search theory was conducted during 1985 and was supported in part by U.S. Army contract No. MDA 903-

85-C-0012. These efforts were assisted by the Center for Naval Analyses. We particularly want to thank H. R. Richardson for his guidance, attention, and help. Members of the Columbia Applied Mathematics Group, and the editors in particular, would like to express their deepest gratitude to W. Brown for his constant attention, support, and encouragement in all the efforts of the group. We thank S. Bryen, F. Kapper, S. Lindstrom, and R. Perle of the Department of Defense for their interest and help. Our conversation with N. Friedman was very enlightening and all the members of the group treasured their discussions with C. K. Chu. The editors are indebted to M. Friedman, S. Lalley, and K. Prendergast for the pleasure and enrichment in working with them. Our dear friend H. Robbins, old navy man that he is, has contributed a lion's share in this search-theoretic project, doing some exciting mathematics along the way. Our own number-theoretic problems got even murkier as we looked at their search-theoretic interpretation on the computer screen. Our most sincere thanks are due to Marcel Dekker, Inc., and their editors, who survived the pursuit and evasion of the undersigned.

David V. Chudnovsky
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1

Search Theory

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1. INTRODUCTION

Search theory* came into being during World War II with the work of B. O. Koopman and his colleagues in the Antisubmarine Warfare Operations Research Group (ASWORG). ASWORG was directed by P. M. Morse and reported to Admiral Ernest King, Chief of Naval Operations and Commander in Chief, U.S. Fleet. Inspired by Morse, many of the fundamental concepts of search theory, such as sweep width and sweep rate, had been established by the spring of 1942. Since that time, search theory has grown to be a major discipline within the field of operations research. Its applications range from deep-ocean search for submerged objects to deep-space surveillance for artificial satellites.

The reader interested in the origins of search theory should consult Morse [29] and Koopman [24]. Early use of search theory to develop naval tactics is discussed in Morse and Kimball [30]; excerpts from this work appear in [31]. For a modern account of search theory, the reader should consult Stone [41]. This book provides a rigorous de opment of the theory and includes an excellent bibliography and notes on previous research. Washburn's monograph [52] is also recommended.

Since World War II, the principles of search theory have been applied successfully in numerous important operations. These include the 1966 search for a lost H-bomb in the Mediterranean near Palomares,

*This article, in essentially the same form, was originally prepared for the Kotz-Johnson *Encyclopedia of Statistical Sciences*, copyrighted by John Wiley & Sons, Inc.

Spain, the 1968 search for the lost nuclear submarine *Scorpion* near the Azores [36], and the 1974 underwater search for unexploded ordnance during clearance of the Suez Canal. The U.S. Coast Guard employs search theory in its open-ocean search and rescue planning [39]. Search theory is also used in astronomy [47] and in radar search for satellites [34]. Numerous additional applications, including those to industry, medicine, and mineral exploration, are discussed in the proceedings [15] of the 1979 NATO Advanced Research Institute on Search Theory and Applications. Applications to biology are given in [17] and [18], and an application to machine maintenance and inspection is described in [32]. Further references to the literature are provided in the first section. This is followed in the second section by an illustration of how search theory can be used to solve an optimal search problem.

2. REVIEW OF SEARCH THEORY LITERATURE

Work in search theory can be classified, at least in part, according to the assumptions made about measures of effectiveness, target motion, and the way in which search effort is characterized. This chapter is organized according to these criteria.

2.1 Measures of Effectiveness

Among the many measures of effectiveness that are used in search analysis, the most common are:

1. Probability of detection
2. Expected time to detection
3. Probability of correctly estimating target "whereabouts"
4. Entropy of the posterior target location probability distribution

Usually, the objective of an optimal search is to maximize the probability of detection with some constraint imposed on the amount of search effort available. For a stationary target, it is shown in Stone [41] that when the detection function is concave or the search space and search effort are continuous, a plan that maximizes the probability of detection in each of successive increments of search effort (incrementally optimal) will also be optimal for the total effort contained in the increments (totally optimal).

Moreover, for stationary targets, it is often theoretically possible to construct a "uniformly optimal" search plan. This is a plan for which probability of detection is maximized at each moment during its period of application. If a uniformly optimal search plan exists, it will (a) maximize the probability of detection over any period of application (i.e., be totally and incrementally optimal), and (b) minimize the expected time to detection. An example of such a plan (originally due to Koopman) is given in the final section.

In a "whereabouts" search, the objective is to estimate correctly the target's location in a collection of cells given a constraint on search cost. The searcher may succeed either by finding the target during search or by correctly guessing the target's location after search. These searches were first studied systematically by Kadane (see [21]). In many cases of interest, Kadane shows that the optimal whereabouts search consists of an optimal detection search among all cells *exclusive* of the cell with the highest prior target location probability. If the search fails to find the target, one guesses that it is in the excluded highest-probability cell.

More recently, Kadane and Stone [22] have considered whereabouts search in the context of moving targets. They show that the optimal whereabouts search plan may be found by solving a finite number of optimal detection search problems, one for each cell in the grid.

Consideration of entropy as a measure of effectiveness is useful in certain situations and can be used to draw a distinction between search and surveillance. For certain stationary target detection search problems with an exponential detection function, Barker [5] has shown that the search plan that maximizes the entropy of the posterior target location probability distribution conditioned upon search failure is the same as the search plan that maximizes probability of detection.

In a surveillance search, the objectives are usually more complex than in a detection search. For example, one may wish to estimate target location correctly at the end of a period of search in order to take some further action. In this case, detection before the end of the period can contribute to success but does not in itself constitute success. More general problems of this type are discussed by Tierney and Kadane [49], and they obtain necessary conditions for optimality when target motion is Markovian. In surveillance problems involving moving targets and false contacts where the time of terminal action may not be known in advance, Richardson [38] suggests allocating search effort to minimize expected entropy (maximize information).

Among other measures of effectiveness that are used in search, those based on minimax criteria are of particular interest. Corwin [8] considers search as a statistical game and seeks estimates for target location. Alpern [3], Gal [14], and Isaacs [19] consider games in which minimax strategies are sought for a moving target seeking to avoid a moving searcher.

2.2 Target Motion

Assumptions about target motion have a considerable influence on the characteristics of search plans and the difficulty of computation. Until recently all but the simplest search problems involving target motion were intractable from the point of view of mathematical optimization. Results were usually obtained by considering transformations that would convert the problem into an equivalent stationary target problem (e.g., see Stone and Richardson [42], Stone [43], and Pursiheimo [35]). Representative early work on search with Markovian target motion is given in Pollock [33], Dobbie [12], and McCabe [28]. Hellman [16] investigates the effect of search upon targets whose motion is a diffusion process.

The first computationally practical solution to the optimal search problem for stochastic target motion involving a large number of cells and time periods is due to Brown [6]. For exponential detection functions, he found necessary and sufficient conditions for search plans for discrete time and space, and provided an iterative method for optimizing search for targets whose motion is described by mixtures of discrete time and space Markov chains. Washburn [51] extended Brown's necessary conditions to the case of discrete search effort. Washburn [53] also provides a useful bound on how close a plan is to the optimal plan.

Very general treatments of moving target search are provided by Stone [44] and by Stromquist and Stone [46], allowing efficient numerical solution in a wide class of practical moving target problems; these include, for example, non-Markovian motion and nonexponential detection functions.

The existence of optimal search plans for moving targets is not to be taken for granted. L. K. Arnold has shown that there are cases where no allocation function satisfies the necessary conditions given in [46]. In his examples, there appear to be optimal plans, but they concentrate effort on sets of measure zero and are outside the class of search allocation functions usually considered. He also shows the existence of optimal plans whenever the search density is constrained to be bounded.

2.3 Search Effort

Search effort may be either discrete (looks, scans, etc.) or continuous (time, track length, etc.). In problems involving discrete search effort, the target is usually considered to be located in one of several cells or boxes. The search consists of specifying a sequence of looks in the cells. Each cell has a prior probability of containing the target. A detection function b is specified, where b(j,k) is the conditional probability of detecting the target on or before the kth look in cell j, given that the target is located in cell j. A cost function c is also specified, where c(j,k) is the cost of performing k looks in cell j. An early solution to this problem for independent glimpses and uniform cost was given by Chew [7]. In this case, for $0 \le a_j \le 1$,

$$b(j,k) - b(j, k - 1) = a_j(1 - a_j)^{k-1}$$

for all j and for k > 0; c(j,k) = k for all j and k ≥ 0. Additional important results have been obtained by Matula [27], Blackwell (see Matula [27]), Kadane [20], and Wegener [54]-[56].

Kadane's result [20] is particularly interesting since he uses a variant of the Neyman-Pearson lemma to obtain an optimal plan for the general case where b(j,k) - b(j, k - 1) is a decreasing function of k for all j.

In problems involving continuous effort, the target may be located in Euclidean n-space or in cells as in the case of discrete search. In the former case it is assumed that the search effort is "infinitely divisible" in the sense that it may be allocated as finely as necessary over the entire search space. The search problem was originally expressed in this form by Koopman (see [23]). The continuous effort case will be considered in greater detail in the remainder of this chapter.

Just as with discrete search effort, there is a detection function b, where b(x,z) is the probability of detecting the target with z amount of effort applied to the point x, given that the target is located at x. If x is a cell index, then z represents the amount of time or track length allocated to the cell. If x is a point in Euclidean n-space, then z is a density, as will be made clear in the next section.

Koopman's original solution [23] to the search problem made use of an exponential function for b of the form

$$b(x,z) = 1 - \exp(-\kappa z),$$

where κ is a positive constant that may depend on x. DeGuenin [9] considered a more general class of detection functions now referred to as "regular" (see [41]). Dobbie [11] considered sequential search with a concave detection function. Richardson and Belkin [37] have treated a special type of regular detection function obtained when the parameter κ in the exponential effectiveness function is a random variable. Such functions occur when sensor capabiliti s are uncertain. Tatsuno and Kisi [48] address similar problems. Stone has considered very general detection functions and has collected the results in [41]. For differentiable detection functions, Wagner [50] obtained sufficient conditions for an optimal search plan with continuous effort using a nonlinear functional version of the Neyman-Pearson lemma.

2.4 Remarks

Search theory remains a field of active research despite the considerable advances made since its inception more than 40 years ago. A review of the current status of search theory in terms of practical applications is given in [45]. Many problems remain to be solved, particularly in cases involving multiple targets and false targets. Also, systematic methods are needed for constructing the prior target location probability distribution from sometimes conflicting subjective opinion. More work is also needed on problems where it is essential to take exact account of search track continuity or the switching cost of moving from one region to another. These problems remain intractable, although some recent progress has been made (see, e.g., [54] and [25]).

Brown's innovative solution to an important class of moving target search problems has removed an impediment to progress in this area.