A Handbook of

TRIGONOMETRIC FUNCTIONS -Introducing Doversines

Leon Kennedy

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Trigonometric Functions—Introducing Doversines

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Preface

THE MAJOR PURPOSE of this book is to supply the user with a table of trigonometric functions that is arranged specifically to expedite the solution of both plane and spherical triangles. For any one degree from 0° to 180° the user has in full view the natural sines, cosines, tangents, cotangents, secants, cosecants, doversines, and haversines, as well as the logarithms of these various functions. The natural functions and their logarithms have been printed in black and the logarithms, located in adjacent columns, have been printed in blue. Instead of having the functions arranged in the conventional semi-quadrant manner, there are two separate pages for *each* consecutive degree from 0° to 180°.

The arrangement presented in this book eliminates certain inconveniences encountered when one uses a trigonometric table that is arranged in the conventional semi-quadrant manner. The arguments and functions are found at locations that are consistent for each degree throughout the table presented in this book.

The natural trigonometric functions have been calculated to either six significant figures or to six decimal places, whichever case yields a minimum number of digits for the function. The logarithms of the functions have been calculated to six decimal places. A table of six place logarithms of the numbers 1 to 10000 has been included at the end of the trigonometric function table.

The haversine (one-half of the versine) was used extensively in the nineteenth century and the early twentieth century by marine navigators. The *"law of haversines"* was used to solve the spherical oblique triangle encountered in celestial navigation. The *"law of haversines"* is a simplification of the law of cosines for the spherical triangle.

Today, the celestial navigator no longer has to solve the spherical oblique triangle in a formal manner. The triangle has been solved and the solutions tabulated as a function of the variables involved in the celestial navigation problem.[1] As a consequence of the modern methods of celestial navigation, accurate haversine tables are not found as readily today as they were earlier. The haversines are included in this book since they do have a distinct advantage when one is called upon to solve the spherical oblique triangle for a numerical answer.

It is the author's opinion that the haversine method for solving the spherical triangle could be used to an advantage by a marine navigator engaged in off-shore cruising (or sailing) aboard a small craft in which space may be limited. The haversine method of celestial navigation requires only one small book of trigonometric functions, whereas some of the modern methods require several volumes.[2]

A trigonometric function which simplifies the law of cosines for the plane oblique triangle has been defined and tabulated in this book. The new function is called the doversine (doubled versine). The doversines and their logarithms were first computed by means of the Illiac digital computer. A second computation using an IBM 650 computer provided an excellent check for any errors that might have occurred during the calculations.

A summary of trigonometric relations that are pertinent to the solution of both plane and sperical triangles is given in the Introduction. Both the professional person and the student will find the arrangement of tables presented in this book workable and convenient for the routine solution of both plane and sperical triangles.

The author expresses his appreciation to Dr. J. P. Nash of the University of Illinois for having the Illiac digital computer programmed to calculate the doversines; to Mr. Russell Altenberger of the Iowa State University for many valuable suggestions concerning the IBM 650 computer and associated IBM equipment; to Professor Joseph Senne of the Iowa State University for encouraging the publication of the doversines; to the Iowa State University Press through which publication has been realized; and to my wife who helped with the tedious job of checking the computer output for errors.

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[1] For example, the U. S. Hydrographic Office publication *H. O. 214*.

[2] This does not apply to the Ageton method of celestial navigation. The Ageton method is contained in one small volume, *Manual of Celestial Navigation*, by Arthur A. Ageton, D. Van Nostrand Company, Inc., New York, 1952.

Introduction

TRIGONOMETRIC FORMULAS that are necessary for the solution of both plane and spherical triangles are summarized at the end of this section. Since the versine function is encountered only rarely, a short discussion of the *"law of haversines"* and the *"law of doversines"* is in order.

The *"law of haversines"* is derived from the law of cosines for the spherical triangle in the following manner:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Since $\text{vers}\ x = 1 - \cos x,$

$$1 - \text{vers}\ a = \cos b \cos c + \sin b \sin c\ (1 - \text{vers}\ A),$$
$$= \cos(b - c) - \sin b \sin c\ \text{vers}\ A,$$
$$= 1 - \text{vers}(b - c) - \sin b \sin c\ \text{vers}\ A.$$

Therefore, $\text{vers}\ a = \text{vers}(b - c) + \sin b \sin c\ \text{vers}\ A.$

Since $\text{hav}\ x = \frac{1}{2}\ \text{vers}\ x,$

$$\text{hav}\ a = \text{hav}(b - c) + \sin b \sin c\ \text{hav}\ A.$$

The haversine function is symmetrical about zero degrees, i.e., $\text{hav}\ 30° = \text{hav}(-30°)$. Therefore, it makes no difference whether or not the quantity $(b - c)$ is positive or negative. Many books suggest that this difference is taken by always subtracting the smaller of the two quantities from the larger. A notation that is sometimes used to represent this operation is $(b \sim c)$. Since the symbol $(\sim)$ has several other meanings in mathematics, it has been decided to use a different notation to represent this operation. **The notation employed in this book will be d_{bc}.**

The notation d_{bc} is defined as meaning the difference of the sides b and c obtained by subtracting the smaller side from the larger side.

The *"law of haversines"* for the spherical oblique triangle is:

$$\text{hav } a = \text{hav}(d_{bc}) + \sin b \sin c \text{ hav } A.$$

Cyclic changes of the letters in the above expression yield the other two forms of the *"law of haversines"*:

$$\text{hav } b = \text{hav}(d_{ca}) + \sin c \sin a \text{ hav } B,$$

and

$$\text{hav } c = \text{hav}(d_{ab}) + \sin a \sin b \text{ hav } C.$$

If the three angular sides of a spherical triangle are known and it is desired to compute one angle, the *"law of haversines"* may be used in the following form:

$$A = \text{Hav}^{-1}\left[\csc b \csc c \sqrt{\text{hav}(a + d_{bc}) \text{ hav}(a - d_{bc})}\right].$$

The other two forms of this expression may also be obtained by cyclic changes of the letters. These expressions for one of the angles in terms of the sides may be solved quite easily by the use of logarithms.

It should be pointed out that since the haversines are positive for *all* angles (and the sines and cosecants are positive in the first and second quadrants), the signs for the terms in the previous expressions cannot change as the angles vary from 0° to 180°. This is one advantage that the *"law of haversines"* has over the law of cosines for the spherical triangle.

The *"law of doversines"* is derived from the law of cosines for the plane triangle in the following manner:

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

Adding 2bc to each side of the law of cosines gives

$$a^2 = b^2 + c^2 - 2bc \cos A + 2bc - 2bc.$$

Factoring, $a^2 = (b - c)^2 + 2bc(1 - \cos A)$.

Now, $2(1 - \cos x) = 2 \text{ vers } x = \text{dov } x$.

Therefore, $a^2 = (b - c)^2 + bc \text{ dov } A$.

Employing the notation d_{bc} as previously defined, the *"law of doversines"* for the plane triangle is:

$$a^2 = d^2_{bc} + bc \text{ dov } A.$$

Cyclic changes of the letters in the previous expression yield the other two forms of the "*law of doversines*":

$$b^2 = d^2_{ca} + ca \text{ dov } B,$$

and

$$c^2 = d^2_{ab} + ab \text{ dov } C.$$

The "*law of doversines*" may be stated as follows:

In any plane triangle, the square of one side is equal to the difference of the other two sides squared, plus the product of the latter two sides and the doversine of their included angle.

There are several advantages that the "*law of doversines*" has over the law of cosines for the plane triangle. First, like the haversines, the doversine function is positive for all angles. Therefore, the term *bc dov A* is always added to d^2_{bc}. The "*law of doversines*" eliminates one squaring operation. Since the *2* is defined as part of the trigonometric function, there is one less step in calculating. The "*law of doversines*" is very convenient to use in a situation where none of the angles are known, all three sides of a plane triangle are given, and it is necessary to calculate one of the angles. Solving the "*law of doversines*" for A, we obtain:

$$A = \text{Dov}^{-1}\left[\frac{(a + d_{bc})\ (a - d_{bc})}{bc}\right].$$

Cyclic changes of the letters in the above expression yield the other two forms:

$$B = \text{Dov}^{-1}\left[\frac{(b + d_{ca})\ (b - d_{ca})}{ca}\right],$$

and

$$C = \text{Dov}^{-1}\left[\frac{(c + d_{ab})\ (c - d_{ab})}{ab}\right].$$

Unlike the expressions obtained by solving the law of cosines for one angle, the preceding expressions can readily be solved by the use of logarithms.

A formula that is frequently used in place of the law of cosines for an explicit expression of an angle in terms of the sides is:

$$A = 2 \, \mathrm{Tan}^{-1} \sqrt{\frac{(s - c)(s - b)}{s(s - a)}} ,$$

where $s = \tfrac{1}{2}(a + b + c)$.

Clearly, the expression involving doversines is easier to evaluate than the above tangent formula.

The tables presented in this book are self-explanatory. It is felt that the people using this book of tables will be familiar with the methods of interpolation. If this is not the case, the reader should consult a trigonometry textbook.

Quadrant	sin *x*	cos *x*	tan *x*	cot *x*	sec *x*	csc *x*	dov *x*	hav *x*
I	+	+	+	+	+	+	+	+
II	+	−	−	−	−	+	+	+
III	−	−	+	+	−	−	+	+
IV	−	+	−	−	+	−	+	+

Fig. 1 — Signs of the trigonometric functions from 0° to 360°

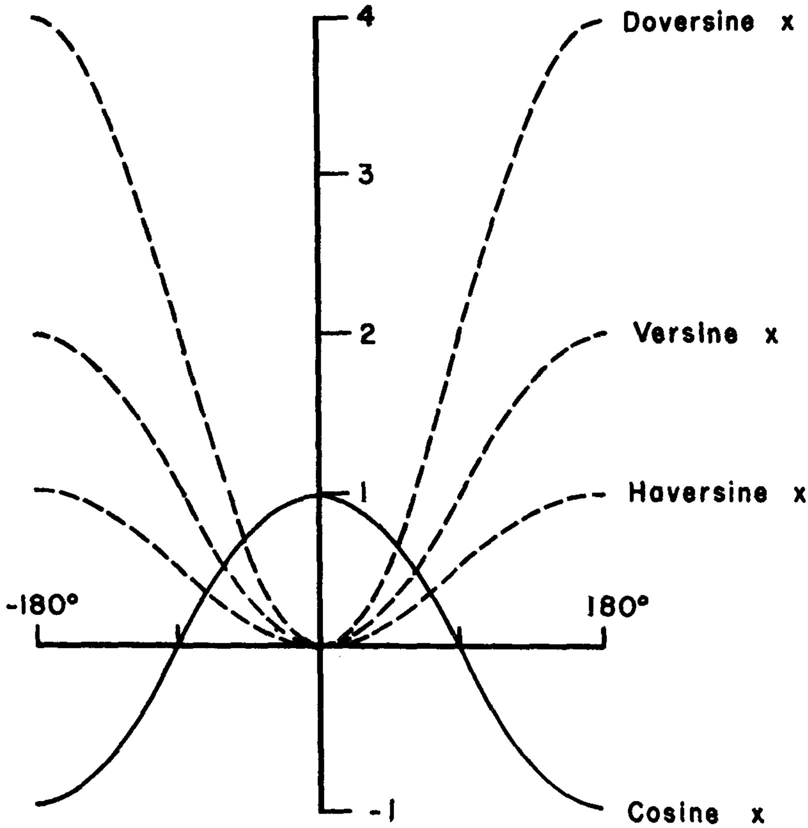


Fig. 2 — A comparison of the versine, doversine, and haversine functions with the cosine function.

Basic Trigonometric Formulas and Identities

Fundamental identities:

$$\csc x = 1/\sin x, \qquad \sec x = 1/\cos x,$$
$$\tan x = \sin x/\cos x, \qquad \cot x = 1/\tan x,$$
$$\cot x = \cos x/\sin x, \qquad \sin^2 x + \cos^2 x = 1,$$
$$1 + \tan^2 x = \sec^2 x, \qquad 1 + \cot^2 x = \csc^2 x$$

Reduction formulas:

$$\sin (90° \pm x) = \cos x, \qquad \cos (90° \pm x) = \mp \sin x,$$
$$\tan (90° \pm x) = \mp \cot x, \qquad \cot (90° \pm x) = \mp \tan x,$$
$$\sec (90° \pm x) = \mp \csc x, \qquad \csc (90° \pm x) = \sec x.$$

$$\sin (180° \pm x) = \mp \sin x, \qquad \cos (180° \pm x) = - \cos x,$$
$$\tan (180° \pm x) = \pm \tan x, \qquad \cot (180° \pm x) = \pm \cot x,$$
$$\sec (180° \pm x) = - \sec x, \qquad \csc (180° \pm x) = \mp \csc x.$$

$$\sin (270° \pm x) = - \cos x, \qquad \cos (270° \pm x) = \pm \sin x,$$
$$\tan (270° \pm x) = \mp \cot x, \qquad \cot (270° \pm x) = \mp \tan x,$$
$$\sec (270° \pm x) = \pm \csc x, \qquad \csc (270° \pm x) = - \sec x.$$

$$\sin(-x) = -\sin x, \qquad \cos (-x) = \cos x,$$
$$\tan(-x) = -\tan x, \qquad \cot (-x) = - \cot x,$$
$$\sec(-x) = \sec x, \qquad \csc (-x) = - \csc x.$$

Sum formulas:

$$\sin x + \sin y = 2 \sin \tfrac{1}{2} (x+y) \cos \tfrac{1}{2} (x - y),$$
$$\sin x - \sin y = 2 \cos \tfrac{1}{2} (x+y) \sin \tfrac{1}{2} (x - y),$$
$$\cos x + \cos y = 2 \cos \tfrac{1}{2} (x+y) \cos \tfrac{1}{2} (x - y),$$
$$\cos x - \cos y = - 2 \sin \tfrac{1}{2} (x+y) \sin \tfrac{1}{2} (x - y).$$

Formulas for the sum and difference of two angles:

$$\sin (x \pm y) = \sin x \cos y \pm \cos x \sin y,$$
$$\cos (x \pm y) = \cos x \cos y \mp \sin x \sin y,$$
$$\tan (x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}.$$

Double-angle formulas:

$$\sin 2x = 2 \sin x \cos x, \qquad \cos 2x = \cos^2 x - \sin^2 x,$$
$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}.$$

Half-angle formulas:

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}, \quad \cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}},$$

$$\tan\frac{x}{2} = \frac{\sin x}{1+\cos x}.$$

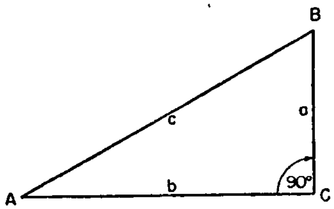
Product formulas:

$$\sin x \sin y = \tfrac{1}{2}\cos(x-y) - \tfrac{1}{2}\cos(x+y),$$
$$\sin x \cos y = \tfrac{1}{2}\sin(x-y) + \tfrac{1}{2}\sin(x+y),$$
$$\cos x \cos y = \tfrac{1}{2}\cos(x-y) + \tfrac{1}{2}\cos(x+y).$$

Miscellaneous formulas:

$$\begin{aligned}
\text{versine } x &= \text{vers } x &&= 1 - \cos x\\
\text{haversine } x &= \text{hav } x &&= \tfrac{1}{2}\text{ vers } x\\
\text{doversine } x &= \text{dov } x &&= 2 \text{ vers } x\\
\text{coversine } x &= \text{covers } x &&= 1 - \sin x\\
\text{exsecant } x &= \text{exsec } x &&= (\sec x - 1
\end{aligned}$$

SOLUTION OF PLANE RIGHT TRIANGLES



The right triangle can be solved if at least one side and one angle (in addition to the 90° angle) are known, or if at least two sides are known.

EXAMPLES:

Given b, c:

$$A = \mathrm{Cos}^{-1}\left(\frac{b}{c}\right)$$
$$B = 90° - A$$
$$a = b \tan A$$

Given a, c:

$$B = \mathrm{Cos}^{-1}\left(\frac{a}{c}\right)$$
$$A = 90° - B$$
$$b = c \cos A$$

Given a, b:

$$A = \mathrm{Tan}^{-1}\left(\frac{a}{b}\right)$$
$$B = 90° - A$$
$$c = b \sec A$$

Given A, a:

$$B = 90° - A$$
$$b = a \cot A$$
$$c = a \sec B$$

Given A, b:

$$B = 90° - A$$
$$a = b \tan A$$
$$c = b \sec A$$

Given A, c:

$$B = 90° - A$$
$$a = c \cos B$$
$$b = c \cos A$$

Given B, a:

$$A = 90° - B$$
$$b = a \cot A$$
$$c = a \sec B$$

Given B, b:

$$A = 90° - B$$
$$a = b \tan A$$
$$c = b \sec A$$

Given B, c:

$$A = 90° - B$$
$$a = c \cos B$$
$$b = c \cos A$$

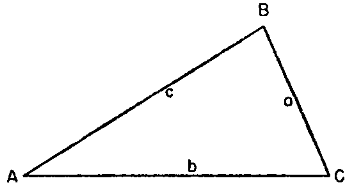
SOLUTION OF PLANE OBLIQUE TRIANGLES

Law of sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

"Law of doversines":

$$a^2 = d_{bc}^2 + bc \text{ dov } A,$$
$$b^2 = d_{ca}^2 + ca \text{ dov } B,$$
$$c^2 = d_{ab}^2 + ab \text{ dov } C.$$



The minimum data required to solve the plane oblique triangle may be classified into four distinct cases. These four cases are as follows:

CASE I. Given all three sides.

CASE II. Given any two sides and their included angle.

CASE III. Given any two angles and any one side.

CASE IV. Given any two sides and an angle opposite one of the sides.

An example of each case is:

CASE I, given a, b, c.

$$A = \text{Dov}^{-1}\left[\frac{(a + d_{bc})\,(a - d_{bc})}{bc}\right],\ B = \text{Sin}^{-1}\left[\frac{b}{a}\sin A\right],$$
$$C = 180° - (A + B).$$

CASE II, given C, a, b.

$$c = \sqrt{d_{ab}^2 + ab \text{ dov } C},\ \ B = \text{Sin}^{-1}\left[\frac{b}{c}\sin C\right],$$
$$A = 180° - (B + C).$$

CASE III, given A, B, a.

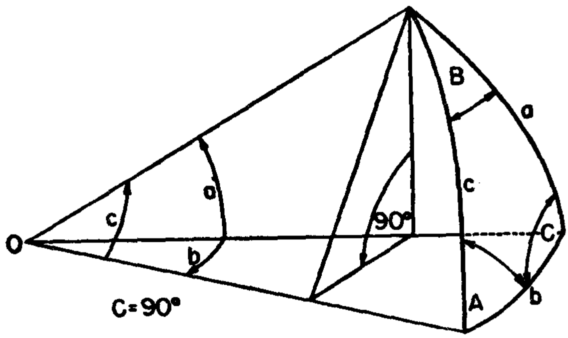
$$C = 180° - (A + B),\ \ b = a \sin B \csc A,\ \ c = a \sin C \csc A.$$

CASE IV, given A, a, b.

$$B = \text{Sin}^{-1}\left[\frac{b}{a}\sin A\right],\ \ C = 180° - (A + B),\ \ c = a \sin C \csc A.$$

A rough sketch of the triangle to be solved will be very useful before beginning the solution. The sketch need only be accurate enough to tell whether the angles sought are obtuse or acute. Case IV is sometimes called the ambiguous case since there may be two solutions to the triangle.

SOLUTION OF SPHERICAL RIGHT TRIANGLES

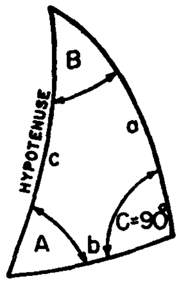


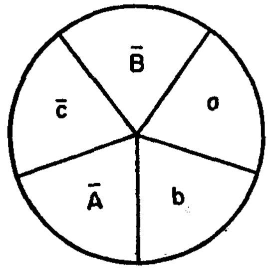
The spherical triangle (right or oblique) differs from the plane triangle in the following ways:

1.) The sides of the spherical triangle are segments of great circles and described in terms of angles rather than length. Therefore, a solution of the spherical triangle gives no information as to the physical size of the triangle. The curvature (or radius of curvature) must be known in order to ascertain the length of the sides and the area of a spherical triangle.
2.) The sum of the three sides of a spherical triangle is less than 360°; i. e., $a + b + c < 360°$.
3.) The angles of a spherical triangle are dihedral angles of the planes defined by the sides. The expression $A + B + C = 180°$ *does not* apply to the spherical triangle. The sum of the angles of a spherical triangle is greater than 180° and less than 540°; i. e., $180° < A + B + C < 540°$.

The spherical right triangle can be solved if at least two quantities are known in addition to the 90° angle.[1] Ten formulas are available for the solution of spherical right triangles. John Napier (1550 – 1617) invented a device known as Napier's Wheel in order to help one remember these ten formulas. Construct Napier's Wheel as shown being careful to keep the quantities in the same order as they appear in the spherical triangle. The 90° angle (C) is omitted. The symbol $\bar{A}$ means the complement of A; i. e., $90° - A$.

[1] Two solutions exist when the given parts are a side and the angle opposite.





NAPIER'S WHEEL

NAPIER'S RULE I. Referring to Napier's Wheel, the sine of any part is equal to the product of the cosines of the opposite parts.

NAPIER'S RULE II. The sine of any part is equal to the product of the tangents of the adjacent parts.

For example:

$$\sin \bar{A} = \cos \bar{B} \cos a.$$

Since $\sin \bar{A} = \cos A$, and $\cos \bar{B} = \sin B$,

then $\cos A = \sin B \cos a.$ (1)

$$\sin \bar{A} = \tan \bar{c} \tan b.$$

Since $\sin \bar{A} = \cos A$, and $\tan \bar{c} = \cot c$,

then $\cos A = \cot c \tan b.$ (2)

The following eight formulas could be found in a like manner:

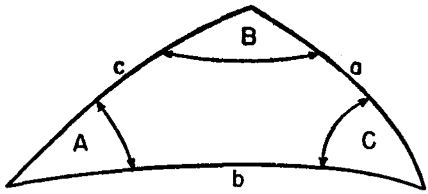
$\sin b = \sin c \sin B$, $\cos c = \cos a \cos b$, $\cos c = \cot A \cot B$,
$\cos B = \tan a \cot c$, $\sin a = \sin c \sin A$, $\cos B = \cos b \sin A$.
$\sin a = \tan b \cot B$, $\sin b = \tan a \cot A$,

The following propositions from solid geometry are quite useful:

I. A side and angle opposite it are always of the same quadrant.

II. If the hypotenuse is less than 90°, the sides are of the same quadrant. If the hypotenuse is greater than 90°, the sides are of opposite quadrants.

SOLUTION OF SPHERICAL OBLIQUE TRIANGLES



The minimum data required to solve the spherical oblique triangle may be classified into six distinct cases:

CASE I. Given three sides.

CASE II. Given two sides and the included angle.

CASE III. Given three angles.

CASE IV. Given two angles and the included side.

CASE V. Given two sides and an angle opposite one of these sides.

CASE VI. Given two angles and a side opposite one of these angles.

Cases V and VI are known as the ambiguous cases. There may be no solution, a single solution, or two solutions.

The following expressions may be used to solve the spherical oblique triangle:

"Law of haversines":

$$\text{hav } a = \text{hav}(d_{bc}) + \sin b \sin c \text{ hav } A,$$

$$A = \text{Hav}^{-1} \left[\csc b \csc c \sqrt{\text{hav } (a + d_{bc}) \text{ hav } (a - d_{bc}} \right],$$

$$\text{hav } b = \text{hav}(d_{ca}) + \sin c \sin a \text{ hav } B,$$

$$B = \text{Hav}^{-1} \left[\csc c \csc a \sqrt{\text{hav}(b + d_{ca}) \text{ hav}(b - d_{ca})} \right],$$

$$\text{hav } c = \text{hav}(d_{ab}) + \sin a \sin b \text{ hav } C,$$

$$C = \text{Hav}^{-1} \left[\csc a \csc b \sqrt{\text{hav}(c + d_{ab}) \text{ hav}(c - d_{ab})} \right].$$

Law of sines:

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

Gauss's formulas:

$$\sin \tfrac{1}{2}(A + B) = \frac{\cos \tfrac{1}{2}(a - b)}{\cos \tfrac{1}{2}c} \cos \tfrac{1}{2}C,$$

$$\sin \tfrac{1}{2}(A - B) = \frac{\sin \tfrac{1}{2}(a - b)}{\sin \tfrac{1}{2}c} \cos \tfrac{1}{2}C,$$

$$\cos \tfrac{1}{2}(A + B) = \frac{\cos \tfrac{1}{2}(a + b)}{\cos \tfrac{1}{2}c} \sin \tfrac{1}{2}C,$$

$$\cos \tfrac{1}{2}(A - B) = \frac{\sin \tfrac{1}{2}(a + b)}{\sin \tfrac{1}{2}c} \sin \tfrac{1}{2}C.$$

Napier's analogies:

$$\frac{\sin \tfrac{1}{2}(A - B)}{\sin \tfrac{1}{2}(A + B)} = \frac{\tan \tfrac{1}{2}(a - b)}{\tan \tfrac{1}{2}c},$$

$$\frac{\sin \tfrac{1}{2}(a - b)}{\sin \tfrac{1}{2}(a + b)} = \frac{\tan \tfrac{1}{2}(A - B)}{\cot \tfrac{1}{2}C},$$

$$\frac{\cos \tfrac{1}{2}(A - B)}{\cos \tfrac{1}{2}(A + B)} = \frac{\tan \tfrac{1}{2}(a + b)}{\tan \tfrac{1}{2}c},$$

$$\frac{\cos \tfrac{1}{2}(a - b)}{\cos \tfrac{1}{2}(a + b)} = \frac{\tan \tfrac{1}{2}(A + B)}{\cot \tfrac{1}{2}C}.$$

The following facts may be useful:

I. *The order of magnitude of the sides of a spherical triangle is the same as the order of magnitude of the respective angles, i. e., if* $a > c > b$, *then* $A > C > B$.

II. *The sum of two sides of a spherical triangle is greater than the third side.*

III. *Half the sum of any two sides is of the same quadrant as half the sum of the opposite angles.*