AXIOMATIC SET THEORY

P. BERNAYS

A. A. FRAENKEL

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PAUL BERNAYS

Professor of Mathematics
Eidg. Techn. Hochschule, Zürich

WITH A HISTORICAL INTRODUCTION

by

ABRAHAM A. FRAENKEL

Professor of Mathematics
Hebrew University, Jerusalem

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PREFACE

This monograph is designed for a reader who has some acquaintance with problems of axiomatics and with the standard methods of mathematical logic. No special knowledge of set theory and its axiomatics is presupposed.

The Part of Professor Fraenkel gives an introduction to the original Zermelo–Fraenkel form of set-theoretic axiomatics and an account of its following development.

My part is an independent presentation of a formal system of axiomatic set theory. The formal development is carried out in detail, only in chapt. VII, which is about the applications to usual mathematics, it seemed necessary to restrict myself to some indications of the method of englobing analysis, cardinal arithmetic and abstract algebraic theories in the system. These indications, however, certainly will be sufficient to make appear the possibility of such an englobing.

In composing my part I had the continual and most efficient help of Dr. Gert Müller, with whom I have talked over all details. I express him my very hearty thanks.

To North-Holland Publishing Company and its Director Mr. M. D. Frank I am thankful for the obligingness in the technical questions and the elegant accomplishment of the rather complicated print.

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PART I

HISTORICAL INTRODUCTION

BY

A. A. FRAENKEL

HISTORICAL INTRODUCTION

1. Introductory remarks

The axiomatic method in mathematics, which started with Euclid's *Elements* and was revived in the 19th century, again chiefly for the purpose of geometry, has made an enormous progress since the beginning of the 20th century; almost all fields of mathematics and logic, and some of physics and other sciences, have since undergone an axiomatic analysis.

While the axiomatic method is appropriate to the homogeneous and continuous domain of geometry to a greater extent than to arithmetic (where a constructive development from simple objects to complicated ones is natural) in set-theory the axiomatic point of view is particularly appropriate for two reasons. First, the antinomies of set-theory which appeared about the turn of the 20th century, show that the quasi-constructive procedure [1]) of Cantor's set-theory has to be restricted in some way, and thus an axiomatic determination of the restriction becomes imperative. Secondly, the fact that all other mathematical branches can be incorporated in set-theory, leads to the idea of setting up a comprehensive axiom system [2]) of set-theory in which the axiomatic theories of other disciplines can be embedded.

It is natural that, after the shock of the antinomies, the stress should be laid on restricting the concept of set axiomatically in such a way that the known contradictions were eliminated and new ones were not to be expected. This was the trend of Zermelo and his followers (since 1908), as described in Nos. 2–6 below. After confidence in the intrinsic soundness of the theory had been re-established by the success of this step, the question arose whether the restric-

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[1]) A quite different constructive theory, developed by L. E. J. Brouwer since 1907 in accordance with the principles of neo-intuitionism, is outside the subject of the present monograph. See Fraenkel–Bar Hillel [1958], ch. IV.

[2]) A different comprehensive system has been set up in *Principia Mathematica*.

tions imposed on the extent of the set-concept were not exaggerated. Therefore one endeavoured to approach the exact borderline that separates the legitimate theory from the zone of contradictions; this tendency is exhibited in more recent researches (see No. 7 below, and the main part of this monograph [1])).

A few other early attempts to found set-theory axiomatically have either not had sufficient success or not been developed to a point which allows a final judgment [2]).

While a discussion of the axiomatic method in general is beyond the scope of this monograph, a few informal explanations with special reference to set-theory are required. The axiomatization of set-theory renounces a *definition* of the concept of set and of the relation between a set *s* and its elements. The latter, a dyadic

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1) Cf. Borgers [1949] and the comparative surveys of Zermelo's and other methods in Wang [1949] and [1950] and Wang–McNaughton [1953]. (For all references, see the *Bibliography* at the end of the monograph.)

2) Schoenflies [1921] takes the relation between whole and part (proper subset) as the primitive relation. This procedure at best attains a theory of *magnitude* which does not provide for the properties of irreducible parts (elements; see Merzbach [1925]); this also applies to finite sets. — The idea of replacing the element-set relation by the part-whole relation is also the basis of Foradori [1932].

The system of Finsler ([1926], [1933]; Gonseth [1941], pp. 162–180) is based on three axioms only. While by means of the first two axioms it can be proved (see Baer [1928]) that any consistent model of Finsler's system admits of a further extension—as does, for instance, Hilbert's system of geometrical axioms when the axiom of Archimedes is dropped—the *third* axiom postulates completeness in a sense analogous to Hilbert's axiom. For this reason, as well as for the doubts connected with Finsler's notion *zirkelfrei*, his system is hardly tenable.

The axiomatic system of Gonseth [1933] (cf. [1936]) denies the assumption that, given a set, it is settled whether a given object belongs to the set or not. Hence the fundamental propositions on the non-equivalence of sets forfeit their validity and it is premature to judge the difficulties involved.

The intention of Ting–Ho [1938] is similar to that of Zermelo, but the treatement is not strict enough to allow comparison. Cf. also Giorgi [1941]. Systems of a "logicistic" type, such as Ramsey [1926], Quine [1937] and [1940] (cf. [1941] and [1942]), Wang [1954], are not included in the subject-matter of this monograph; neither Lorenzen [1955].

relation (or predicate), is denoted by ε; $x \varepsilon s$ reads "x is contained in, is an element of, belongs to, the set s" or "s contains (the element) x", and its negation is $x \bar{\varepsilon} s$. ε enters as an (undefined) *primitive* relation, *the membership relation.* It is unsymmetrical and the values of its second argument, possibly with the addition of the null-set (see below), constitute the domain of *sets.* Certain statements containing the membership relation and relations defined through it will be introduced as *axioms.* A statement is *true* if and only if it can be deduced from the axioms (by means of a suitable system of logic, in particular certain rules of inference), and the same applies to the *existence* of sets.

The situation is still simpler in the modification **Z** of Zermelo's system given in Nos. 2–6 below; here – as opposed to Zermelo's own system – no other objects than sets appear, hence the first and the second arguments of the membership relation determine the same domain. On the other hand, in the systems briefly mentioned in No. 7 and extensively treated in the main part of this monograph, the second arguments of the membership relation may belong to a different domain, the domain of *classes.*

As to the elimination of contradictions, all that can be expected is the exclusion of the logical and semantical antinomies known at present; this is attained by the exclusion of "overcomprehensive" sets in the system **Z** (cf. No. 7) and by the formulation of Axiom **V** in No. 3. Sufficient hints at how the essential parts of classical set-theory can be derived from the axioms of Nos. 2–5 are given in No. 6. A few remarks only about the independence of axioms occur in this historical part; the question is discussed in a more profound way in the main part of the present monograph.

2. Zermelo's system [1]). Equality and extensionality

The introduction to the main part of this monograph, written by Bernays, begins with a reference to the historically first axiomatization of set-theory, given by Zermelo in 1908. *The chief*

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[1]) The main source is Zermelo [1908a]; see also [1930]. Cf. the expositions in Fraenkel [1927] and [1928] (also Cavaillès [1938], Weyl [1946]), Ackermann [1937], Church [1942] (pp. 180–181).

purpose of this part of the monograph is to give a historical introduction through an exposition of Zermelo's system, with some improvements which were inserted into it before the fundamental changes performed by von Neumann and Bernays from 1925 and 1937 on (see No. 7).

Prior to a systematic exposition, we start with some informal remarks to motivate the method adopted afterwards.

Within a certain non-empty domain of objects we take, as the only primitive relation of our axiomatic system **Z**, the membership relation ε (see above). If x and y denote any objects of the domain, the statement $x \varepsilon y$ shall either hold true or not. While those parts of logic that are necessary for **Z**, in particular the rules of inference and quantification with respect to thing-variables, are assumed to be pre-established, the relation of *equality* should be treated explicitly. Here the following attitudes are possible [1].

1) Equality in its *logical meaning as identity*. Zermelo adopts this attitude by calling x and y equal "if they denote the same thing (object)". When the objects are sets he in addition rests on an axiom of extensionality (see below) which states that a set is determined by its elements.

Thus Zermelo's axiom differs intrinsically from the Axiom of Extensionality as expressed below (p. 8), which refers to *all* objects of the domain [2].

2) Equality as a (second) *primitive relation* within **Z**. Then the usual properties of any equivalence (equality) relation must be guaranteed axiomatically, in particular substitutivity with regard to ε in the two-fold sense: of extensionality as above, and of equal objects being elements of the same sets.

3) Equality as a *mathematically defined relation*. We may define $x=y$ either by "if every set that contains x contains also y and vice versa", or by "if x and y contain the same elements". The

[1]) Cf. Fraenkel [1927] and [1927a], A. Robinso(h)n [1939]. For a more general attitude, cf. Hailperin [1954]. See also the main part of the present monograph.

[2]) The situation becomes somewhat different if, as done in Quine [1940], every "individual" (p. 7) is regarded as a unit-set containing itself.

second way is possible only if, as assumed in the following, every object is a set (including the null-set). In the former case, extensionality must be postulated axiomatically; in the latter, an axiom has to guarantee the former property.

In this part we adopt method 3), which seems superior to 2) insofar as a single primitive relation only occurs in the system, and to 1) since the system is constructed upon a weaker basic discipline. It makes no essential difference which of the two definitions of equality is chosen, provided we take a suitable decision about the existence of *individuals* (called *Urelemente* in Zermelo [1930]), i.e. of objects which contain no element [1]).

Taking into account the admissibility of a set which has no element ("null-set"), three positions about individuals are tenable: that the domain contains one null-set and also other individuals, individuals but no null-set, one null-set but no other individuals. (A domain without null-set and individuals would be impractical.) The first position was taken by Zermelo and, for instance, by Ackermann [1937a]; the second by Quine from 1936 on; the third, first proposed in Fraenkel [1921/22] and later accepted by von Neumann, Bernays, and others, is adopted in the following. This involves that all objects of **Z** are sets, hence that the values of the first and of the second argument of the membership relation constitute the same domain. In fact, for the purpose of developing mathematics it has proved unnecessary to assume the existence of individuals. However, as pointed out below in No. 4, there are problems of independence for which the assumption that infinitely many individuals can be admitted to the domain plays an important part; thus it appears that those problems are more difficult within the system **Z** than in Zermelo's original system.

We now outline the system **Z**, which is not empty and whose only primitive relation is the dyadic relation ε of membership whose arguments are sets.

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[1]) This use of "individual" has nothing to do with the distinction between "individuals" and "classes" in logic (cf., for instance, Tarski [1935], § 2, and the main part of this monograph). In the logical sense the sets are individuals.

Definition I. If s and t are sets such that, for all x, $x \varepsilon s$ implies $x \varepsilon t$, s is called a *subset* of t, in symbols $s \subseteq t$; in particular a *proper subset* ($s \subset t$) if there is a $y \varepsilon t$ with $y \bar{\varepsilon} s$.

In contrast with Cantor's "comprehensive" method of constructing sets, this definition does not allow the construction of subsets of t by "collecting" some of its elements. Only with regard to given sets may we state that one is a subset of the other.

It follows that the relation $\subseteq$ is reflexive ($s \subseteq s$) and transitive (i.e., $s \subseteq t$ and $t \subseteq u$ imply $s \subseteq u$); $\subset$ is irreflexive, transitive, and asymmetrical (i.e., $s \subset t$ and $t \subset s$ are incompatible).

In accordance with earlier remarks, *equality* is defined in either of the following ways.

Definition IIa. If, for all x, $s \varepsilon x$ implies $t \varepsilon x$ and conversely, s equals t ($s = t$); the negation is $s \neq t$ (s differs from t). That is to say, sets are equal if contained in the same objects (sets).

Definition IIb. If $s \subseteq t$ and $t \subseteq s$, then $s = t$; otherwise $s \neq t$. That is to say, sets containing the same objects are equal.

Equality is a reflexive, symmetrical, and transitive relation. The definitions are somehow peculiar to **Z**; in fact, in the systems of No. 7 (below) not every object can become an element of another object, as against IIa, while in Zermelo's own system there may exist different objects without elements, as against IIb.

Equality is substitutive with regard to the second argument of ε, i.e. from $x \varepsilon s$ and $s = t$ it follows that $x \varepsilon t$ [1]). But IIa does not yield extensionality nor does IIb yield substitutivity regarding the first argument; hence we supplement IIa and IIb respectively with the axioms

Axiom **Ia.** $s \subseteq t$ and $t \subseteq s$ imply $s = t$.

Axiom **Ib.** $x \varepsilon s$ and $x = y$ imply $y \varepsilon s$.

It makes no difference whether we adopt Definition IIa and Axiom Ia, or IIb and Ib. Hence we shall simply speak of *the Definition* (II) *of Equality* and of *the Axiom* (I) *of Extensionality.*

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[1]) This is evident in view of IIb; for the proof in view of IIa, cf. A. Robinso(h)n [1939], footnote 4.

Since a set is determined [1]) by its elements, we denote the set with the elements $a, b, c, \ldots$ also by $\{a, b, c, \ldots\}$, regardless of the order of the elements.

Definition III. Two sets without common elements are called *mutually exclusive*. If s contains at least two elements, and any two elements of s are mutually exclusive, s is called a *disjointed* set.

3. "Constructive" axioms of "general" set theory

In the heading two *ad hoc* terms are used. "Constructive" means that, certain things (one set, two sets, a set and a predicate) being given, the axiom states the existence of a *uniquely determined* other set; "general" theory means—in contrast with the use in the main part of the monograph; cf. Bernays [1942a], p. 133—that no axioms of infinity (see No. 5) are included. (According to the present attitude the Axiom of Power-Set, for instance, is not an axiom of infinity.)

The axioms will be preceded by informal remarks which point out the immediate purpose of every single axiom and, thereby, hint at their independence.

The operation of "uniting" two different sets [2]) is introduced by

> *Axiom* II *of Pairing* [3]). For any two different sets a and b, the pair $\{a, b\}$, or $\{b, a\}$, exists.

On account of extensionality we are entitled to use the definite article (*the* pair) here and in the following three axioms, as well as in Theorems 1–3 below.

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[1]) Hence Zermelo's name for Axiom Ia: *Axiom der Bestimmtheit.* Yet he explicitly restricts the axiom to the case that s and t contain elements, a restriction not adopted in our system Z.

[2]) Instead of this Zermelian operation, Kuratowski [1925] uses the union of two sets in the sense of Axiom III. — For the case $a = b$, see theorem 1 on p. 14.

[3]) In Zermelo's terminology, *Axiom der Elementarmengen.* This includes postulating the null-set and the set containing a single given element; both will be *proved* to exist in the present exposition.

Given any number of sets, Axioms I and II do not enable us to produce new sets other than pairs. The primary operations of Boolean algebra, union and intersection, suggest themselves as simplest additional procedures, and the former will prove sufficient. We introduce it by

> *Axiom* III *of Sum-Set (Union).* For any set s which contains at least two elements, there exists the set whose elements are the elements of the elements of s.

This set is called the *sum-set* of s, or the *union of the elements* of s, and is denoted by $\bigcup s$. If $s = \{a, b\}$ we also write $a \cup b$ for $\bigcup s$, and if s contains the elements $a, b, c, \ldots$ we write $a \cup b \cup c \cup \ldots$ for $\bigcup s$.

The union of two different sets exists by II and III. Given a number of sets, certain types of new sets can be produced, i.e. proved to exist, and the associativity of the union-operation is easily shown. Nevertheless, clearly these axioms do not enable us to proceed to more-than-denumerable sets if, say, a sequence of denumerable sets is given, where "denumerable" and "sequence" are informal terms to be formally defined later.

Cantor's (second) tool for reaching higher powers was the operation of (transfinite) multiplication, in particular exponentiation. We can, however, content ourselves with the special tool of the power-set, i.e.

> *Axiom* IV *of Power-Set.* For any set s, there exists the set whose elements are all subsets of s.

This set is called the *power-set* of s and denoted by $\Pi(s)$ [1]).

The efficacy of this axiom differs from that of Cantor's respective operation not only in that the existence of s is presupposed, but in that the existence of the subsets of s is here assumed to be previously established. Since Axioms I–III yield only few very special subsets of a given set and since Definition I does not

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1) Zermelo writes $\mathfrak{S}s$ for $\bigcup s$ (Axiom III), and $\mathfrak{U}s$ for $\Pi(s)$. Thus also, for instance, in Kleene [1952]. In the main part of this monograph $\sum s$ is used for $\bigcup s$.

enable us to produce any subsets [1]), Axiom IV is at the present juncture a very limited instrument and not at all sufficient to yield the so-called "theorem of Cantor" about the cardinality of the power-set. For instance, the existence of *infinite* proper subsets of an infinite set s cannot be ensured so far. Hence methods of producing subsets of a given set remain the chief desideratum – and, as we shall see *a posteriori*, the only one left within general set theory. The principal method in **Z** (for an additional direction see No. 4) is given by

> *Axiom* V *of Subsets* [2]). For any set s and any predicate $\mathfrak{P}$ [3]) which is meaningful ("definite") for all elements of s, there exists the set y that contains just those elements x of s which satisfy the predicate $\mathfrak{P}$ (the condition $\mathfrak{P}(x)$).

y is clearly a subset of s.

The weak point in this formulation of the axiom is the term "meaningful predicate" (or property); in Zermelo's terminology, *definite Eigenschaft.*

Informally this term may be understood to mean that, for each $x \varepsilon s$, $\mathfrak{P}(x)$ should be either true or false, without demanding that the decision ought to be reached at the present stage of scientific development. Thus "x is transcendental" is meaningful when s is a set of numbers, but not "x is finitely definable" or another semantic condition, as those appearing in the antinomies of the semantical type.

Clearly, such explanations cannot satisfy the requirements of a formal deductive theory. Zermelo [1908a] (p. 263) gave the following paraphrase: *Eine Frage oder Aussage* $\mathfrak{E}$, *über deren Gültigkeit oder Ungültigkeit die Grundbeziehungen des Bereiches* [4])

[1]) This is also the case in Zermelo's exposition, but has been misunderstood by some of his interpreters.

[2]) Zermelo calls it *Axiom der Aussonderung* (of "sifting").

[3]) $\mathfrak{P}(x)$ is what is called by Rosser [1953] (e.g., p. 200) *a condition on* x; other expressions are "a statement with free occurrence of x" or "a well-formed formula".

[4]) The intention is to the membership relation, and presumably also to the equality relation.

vermöge der Axiome und der allgemeingültigen logischen Gesetze ohne Willkür entscheiden, heisst "definit". Ebenso wird auch eine "Klassenaussage" 𝔈(x), in welcher der variable Term x alle Individuen einer Klasse 𝔎 durchlaufen kann, als "definit" bezeichnet, wenn sie für jedes einzelne Individuum x der Klasse 𝔎 definit ist. So ist die Frage, ob a ε b oder nicht ist, immer definit, ebenso die Frage, ob M ⊆ N oder nicht.

13 years then elapsed until the first steps were taken to replace this hardly satisfactory explanation by a more rigorous one. Fraenkel [1921/22] and Skolem [1922/23] independently took two seemingly different directions which, however, proved to be essentially equivalent [1]), and of which the second is preferable for its more general and natural character.

The first method formalizes "definiteness" by means of a special concept of function defined by the operations of Axioms **II–V**; the inclusion of V itself leads to a certain hierarchy of orders, in accordance with the fact that the axiom constitutes an axiom schema. The actual derivation of classical (general) set-theory, comprising the theories of order and well-order, shows that this apparently special concept of function is sufficient [2]).

The second method [3]) formalizes "definiteness" by using the concept of (elementary) formula, i.e. of "statement" in the sense of Rosser [1953] (p. 208), obtained from variables and the membership relation by negation, conjunction, disjunction, and quantification with respect to thing-variables, within the first-order predicate calculus with its truth-functions. As proven by Skolem, this procedure covers the first method; it, too, shows the axiom to represent an axiom schema which contains infinitely many particular axioms.

[1]) Cf., in particular, Skolem [1929]. For the first method, cf. Fraenkel [1922] and [1927] and von Neumann [1928a].

[2]) See below No. 6. For existence theorems as those connected with ordinal numbers, the Axiom of Substitution (No. 5), with the generalized function concept of von Neumann [1928a], is required. Cf. also Curry [1934], p. 590, and [1936], p. 375.

[3]) Cf. Skolem [1929] (§ 2) and [1930]; there Schröder's *Algebra der Logik* is used, but this is not an essential feature.