Texts in Applied Mathematics

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Lawrence Perko

Differential Equations and Dynamical Systems

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Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, has led to the establishment of the series: *Texts in Applied Mathematics (TAM)* .

The development of new courses is a natural consequence of a high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos, mix with and reinforce the traditional methods of applied mathematics. Thus, the purpose of this textbook series is to meet the current and future needs of these advances and encourage the teaching of new courses.

TAM will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses, and will complement the *Applied Mathematical Sciences (AMS)* series, which will focus on advanced textbooks and research level monographs.

Preface

This book covers those topics necessary for a clear understanding of the qualitative theory of ordinary differential equations. It is written for upper-division or first-year graduate students. It begins with a study of linear systems of ordinary differential equations, a topic already familiar to the student who has completed a first course in differential equations. An efficient method for solving any linear system of ordinary differential equations is presented in Chapter 1.

The major part of this book is devoted to a study of nonlinear systems of ordinary differential equations. Since most nonlinear differential equations cannot be solved, this book focuses on the qualitative or geometrical theory of nonlinear systems of differential equations originated by Henri Poincaré in his work on differential equations at the end of the nineteenth century. Our primary goal is to describe the qualitative behavior of the solution set of a given system of differential equations. In order to achieve this goal, it is first necessary to develop the local theory for nonlinear systems. This is done in Chapter 2 which includes the fundamental local existence–uniqueness theorem, the Hartman–Grobman Theorem and the Stable Manifold Theorem. These latter two theorems establish that the qualitative behavior of the solution set of a nonlinear system of ordinary differential equations near an equilibrium point is typically the same as the qualitative behavior of the solution set of the corresponding linearized system near the equilibrium point.

After developing the local theory, we turn to the global theory in Chapter 3. This includes a study of limit sets of trajectories and the behavior of trajectories at infinity. Some unsolved problems of current research interest are also presented in Chapter 3. For example, the Poincaré–Bendixson Theorem, established in Chapter 3, describes the limit sets of trajectories of two-dimensional systems; however, the limit sets of trajectories of three-dimensional (and higher dimensional) systems can be much more complicated and establishing the nature of these limit sets is a topic of current research interest in mathematics. In particular, higher dimensional systems can exhibit strange attractors and chaotic dynamics. All of the preliminary material necessary for studying these more advanced topics is contained in this textbook. This book can therefore serve as a springboard for those students interested in continuing their study of ordinary differential equations and dynamical systems. Chapter 3 ends with a technique for constructing the global phase portrait of a two-dimensional dynamical system. The global phase portrait describes the qualitative behavior of the solution set for all time. In general, this is as close as we can come to "solving" nonlinear systems.

In Chapter 4, we study systems of differential equations depending on a parameter. The question of particular interest is: For what values of the parameter does the global phase portrait of a dynamical system change its qualitative structure? The answer to this question forms the subject matter of bifurcation theory. An introduction to bifurcation theory is presented in Chapter 4 where we discuss bifurcations at nonhyperbolic equilibrium points and periodic orbits as well as Hopf bifurcations. Chapter 4 ends with a discussion of homoclinic loop bifurcations for planar systems and an introduction to tangential homoclinic bifurcations and the resulting chaotic dynamics that can occur in higher dimensional systems.

The prerequisites for studying differential equations and dynamical systems using this book are courses in linear algebra and real analysis. For example, the student should know how to find the eigenvalues and eigenvectors of a linear transformation represented by a square matrix and should be familiar with the notion of uniform convergence and related concepts. In using this book, the author hopes that the student will develop an appreciation for just how useful the concepts of linear algebra, real analysis and geometry are in developing the theory of ordinary differential equations and dynamical systems.

I would like to express my sincere appreciation to my colleague Terrence Blows for his many helpful suggestions which led to a substantially improved final version of this book. I would also like to thank Louella Holter for her patience and precision in typing the original manuscript.

Contents

1

Linear Systems

This chapter presents a study of linear systems of ordinary differential equations:

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{1}$$

where $\mathbf{x} \in \mathbf{R}^n$, A is an $n \times n$ matrix and

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix}$$

It is shown that the solution of the linear system (1) together with the initial condition $\mathbf{x}(0) = \mathbf{x}_0$ is given by

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0$$

where e^{At} is an $n \times n$ matrix function defined by its Taylor series. A good portion of this chapter is concerned with the computation of the matrix e^{At} in terms of the eigenvalues and eigenvectors of the square matrix A. Throughout this book all vectors will be written as column vectors and A^T will denote the transpose of the matrix A.

1.1 Uncoupled Linear Systems

The method of separation of variables can be used to solve the first-order linear differential equation

$$\dot{x} = ax.$$

The general solution is given by

$$x(t) = ce^{at}$$

where the constant $c = x(0)$, the value of the function $x(t)$ at time $t = 0$.

Now consider the uncoupled linear system

$$\begin{aligned} \dot{x}_1 &= -x_1 \\ \dot{x}_2 &= 2x_2. \end{aligned}$$

This system can be written in matrix form as

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{1}$$

where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Note that in this case A is a diagonal matrix, $A = \text{diag}[-1, 2]$, and in general whenever A is a diagonal matrix, the system (1) reduces to an uncoupled linear system. The general solution of the above uncoupled linear system can once again be found by the method of separation of variables. It is given by

$$\begin{aligned} x_1(t) &= c_1 e^{-t} \\ x_2(t) &= c_2 e^{2t} \end{aligned} \tag{2}$$

or equivalently by

$$\mathbf{x}(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} \mathbf{c} \tag{2'}$$

where $\mathbf{c} = \mathbf{x}(0)$. Note that the solution curves (2) lie on the algebraic curves $y = k/x^2$ where the constant $k = c_1^2 c_2$. The solution (2) or (2′) defines a motion along these curves; i.e., each point $\mathbf{c} \in \mathbf{R}^2$ moves to the point $\mathbf{x}(t) \in \mathbf{R}^2$ given by (2′) after time t. This motion can be described geometrically by drawing the solution curves (2) in the x_1, x_2 plane, referred to as the *phase plane,* and by using arrows to indicate the direction of the motion along these curves with increasing time t; cf. Figure 1. For $c_1 = c_2 = 0$, $x_1(t) = 0$ and $x_2(t) = 0$ for all $t \in \mathbf{R}$ and the origin is referred to as an *equilibrium point* in this example. Note that solutions starting on the x_1-axis approach the origin as $t \to \infty$ and that solutions starting on the x_2-axis approach the origin as $t \to -\infty$.

The *phase portrait* of a system of differential equations such as (1) with $\mathbf{x} \in \mathbf{R}^n$ is the set of all solution curves of (1) in the phase space $\mathbf{R}^n$. Figure 1 gives a geometrical representation of the phase portrait of the uncoupled linear system considered above. The *dynamical system* defined by the linear system (1) in this example is simply the mapping $\phi: \mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}^2$ defined by the solution $\mathbf{x}(t, \mathbf{c})$ given by (2′); i.e., the dynamical system for this example is given by

$$\phi(t, \mathbf{c}) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} \mathbf{c}.$$

Geometrically, the dynamical system describes the motion of the points in phase space along the solution curves defined by the system of differential equations.

The function

$$\mathbf{f}(\mathbf{x}) = A\mathbf{x}$$

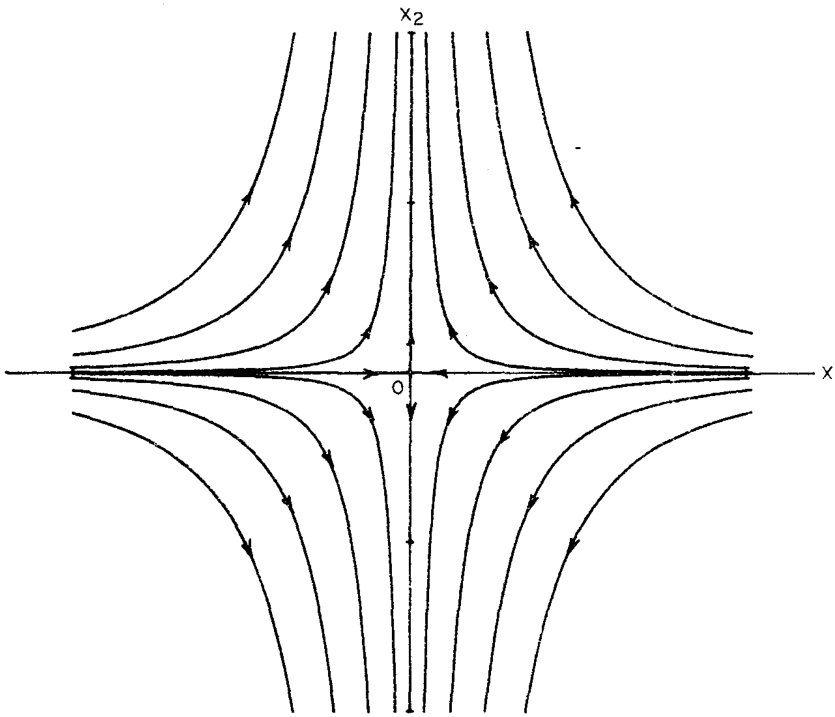

Figure 1

on the right-hand side of (1) defines a mapping $\mathbf{f}: \mathbf{R}^2 \to \mathbf{R}^2$ (linear in this case). This mapping (which need not be linear) defines a *vector field on* $\mathbf{R}^2$; i.e., to each point $\mathbf{x} \in \mathbf{R}^2$, the mapping $\mathbf{f}$ assigns a vector $\mathbf{f}(\mathbf{x})$. If we draw each vector $\mathbf{f}(\mathbf{x})$ with its initial point at the point $\mathbf{x} \in \mathbf{R}^2$, we obtain a geometrical representation of the vector field as shown in Figure 2. Note that at each point $\mathbf{x}$ in the phase space $\mathbf{R}^2$, the solution curves (2) are tangent to the vectors in the vector field $A\mathbf{x}$. This follows since at time $t = t_0$, the velocity vector $\mathbf{v}_0 = \dot{\mathbf{x}}(t_0)$ is tangent to the curve $\mathbf{x} = \mathbf{x}(t)$ at the point $\mathbf{x}_0 = \mathbf{x}(t_0)$ and since $\dot{\mathbf{x}} = A\mathbf{x}$ along the solution curves.

Consider the following uncoupled linear system in $\mathbf{R}^3$:

$$\begin{aligned} \dot{x}_1 &= x_1 \\ \dot{x}_2 &= x_2 \\ \dot{x}_3 &= -x_3 \end{aligned} \tag{3}$$

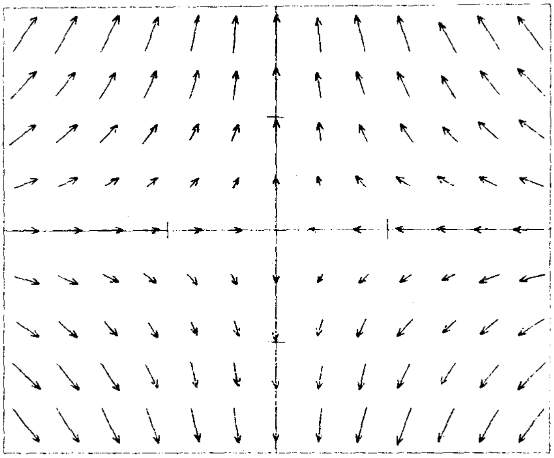

Figure 2

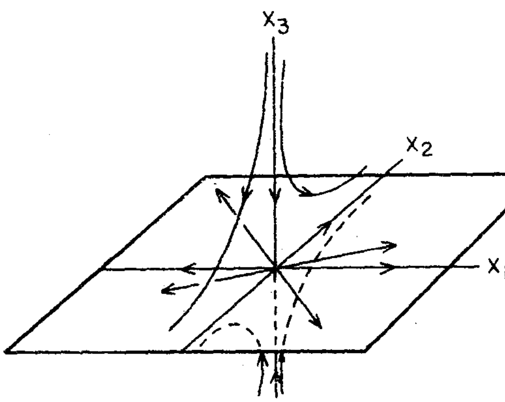

Figure 3

The general solution is given by

$$x_1(t) = c_1 e^t$$
$$x_2(t) = c_2 e^t$$
$$x_3(t) = c_3 e^{-t}.$$

And the phase portrait for this system is shown in Figure 3 above. The x_1, x_2 plane is referred to as the *unstable subspace* of the system (3) and the x_3 axis is called the *stable subspace* of the system (3). Precise definitions of the stable and unstable subspaces of a linear system will be given in the next section.

Problem Set 1

1. Find the general solution and draw the phase portrait for the following linear systems:

(a) $$\begin{aligned}\dot{x}_1 &= x_1\\ \dot{x}_2 &= x_2\end{aligned}$$

(b) $$\begin{aligned}\dot{x}_1 &= x_1\\ \dot{x}_2 &= 2x_2\end{aligned}$$

(c) $$\begin{aligned}\dot{x}_1 &= x_1\\ \dot{x}_2 &= 3x_2\end{aligned}$$

(d) $$\begin{aligned}\dot{x}_1 &= -x_2\\ \dot{x}_2 &= x_1\end{aligned}$$

(e) $$\begin{aligned}\dot{x}_1 &= -x_1 + x_2\\ \dot{x}_2 &= -x_2\end{aligned}$$

Hint: Write (d) as a second-order linear differential equation with constant coefficients, solve it by standard methods, and note that $x_1^2 + x_2^2 =$ constant on the solution curves. In (e), find $x_2(t) = c_2e^{-t}$ and then the x_1-equation becomes a first order linear differential equation.

2. Find the general solution and draw the phase portraits for the following three-dimensional linear systems:

(a) $$\begin{aligned}\dot{x}_1 &= x_1\\ \dot{x}_2 &= x_2\\ \dot{x}_3 &= x_3\end{aligned}$$

(b) $$\begin{aligned}\dot{x}_1 &= -x_1\\ \dot{x}_2 &= -x_2\\ \dot{x}_3 &= x_3\end{aligned}$$

(c) $$\begin{aligned}\dot{x}_1 &= -x_2\\ \dot{x}_2 &= x_1\\ \dot{x}_3 &= -x_3\end{aligned}$$

Hint: In (c), show that the solution curves lie on right circular cylinders perpendicular to the x_1, x_2 plane. Identify the stable and unstable subspaces in (a) and (b). The x_3-axis is the stable subspace in (c) and the x_1, x_2 plane is called the center subspace in (c); cf. Section 1.9.

3. Find the general solution of the linear system

$$\dot{x}_1 = x_1$$
$$\dot{x}_2 = ax_2$$

where a is a constant. Sketch the phase portraits for $a = -1$, $a = 0$ and $a = 1$ and notice that the qualitative structure of the phase portrait is the same for all $a < 0$ as well as for all $a > 0$, but that it changes at the parameter value $a = 0$.

4. Find the general solution of the linear system (1) when A is the $n \times n$ diagonal matrix $A = \text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$. What condition on the eigenvalues $\lambda_1, \ldots, \lambda_n$ will guarantee that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$ for all solutions $\mathbf{x}(t)$ of (1)?

5. What is the relationship between the vector fields defined by

$$\dot{\mathbf{x}} = A\mathbf{x}$$

and

$$\dot{\mathbf{x}} = kA\mathbf{x}$$

where k is a non-zero constant? (Describe this relationship both for k positive and k negative.)

6. (a) If $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are solutions of the linear system (1), prove that for any constants a and b, $\mathbf{w}(t) = a\mathbf{u}(t) + b\mathbf{v}(t)$ is a solution.

   (b) For

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix},$$

   find solutions $\mathbf{u}(t)$ and $\mathbf{v}(t)$ of $\dot{\mathbf{x}} = A\mathbf{x}$ such that every solution is a linear combination of $\mathbf{u}(t)$ and $\mathbf{v}(t)$.

1.2 Diagonalization

The algebraic technique of diagonalizing a square matrix A can be used to reduce the linear system

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{1}$$

to an uncoupled linear system. We first consider the case when A has real, distinct eigenvalues. The following theorem from linear algebra then allows us to solve the linear system (1).

Theorem. *If the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of an $n \times n$ matrix A are real and distinct, then any set of corresponding eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ forms a basis for $\mathbf{R}^n$, the matrix $P = [\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_n]$ is invertible and*

$$P^{-1}AP = \text{diag}[\lambda_1, \ldots, \lambda_n].$$

This theorem says that if a linear transformation $T\colon \mathbf{R}^n \to \mathbf{R}^n$ is represented by the $n \times n$ matrix A with respect to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ for $\mathbf{R}^n$, then with respect to any basis of eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, T is represented by the diagonal matrix of eigenvalues, $\text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$. A proof of this theorem can be found, for example, in Lowenthal [Lo].

In order to reduce the system (1) to an uncoupled linear system using the above theorem, define the linear transformation of coordinates

$$\mathbf{y} = P^{-1}\mathbf{x}$$

where P is the invertible matrix defined in the theorem. Then

$$\mathbf{x} = P\mathbf{y},$$
$$\dot{\mathbf{y}} = P^{-1}\dot{\mathbf{x}} = P^{-1}A\mathbf{x} = P^{-1}AP\mathbf{y}$$

and, according to the above theorem, we obtain the uncoupled linear system

$$\dot{\mathbf{y}} = \text{diag}[\lambda_1, \ldots, \lambda_n]\mathbf{y}.$$

This uncoupled linear system has the solution

$$\mathbf{y}(t) = \text{diag}\left[e^{\lambda_1 t}, \ldots, e^{\lambda_n t}\right]\mathbf{y}(0).$$

(Cf. problem 4 in Problem Set 1.) And then since $\mathbf{y}(0) = P^{-1}\mathbf{x}(0)$ and $\mathbf{x}(t) = P\mathbf{y}(t)$, it follows that (1) has the solution

$$\mathbf{x}(t) = PE(t)P^{-1}\mathbf{x}(0). \tag{2}$$

where $E(t)$ is the diagonal matrix

$$E(t) = \text{diag}\left[e^{\lambda_1 t}, \ldots, e^{\lambda_n t}\right].$$

Corollary. *Under the hypotheses of the above theorem, the solution of the linear system* (1) *is given by the function* $\mathbf{x}(t)$ *defined by* (2).

Example. Consider the linear system

$$\dot{x}_1 = -x_1 - 3x_2$$
$$\dot{x}_2 = 2x_2$$

which can be written in the form (1) with the matrix

$$A = \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 2$. A pair of corresponding eigenvectors is given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The matrix P and its inverse are then given by

$$P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The student should verify that

$$P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Then under the coordinate transformation $\mathbf{y} = P^{-1}\mathbf{x}$, we obtain the uncoupled linear system

$$\begin{aligned} \dot{y}_1 &= -y_1 \\ \dot{y}_2 &= 2y_2 \end{aligned}$$

which has the general solution $y_1(t) = c_1e^{-t}$, $y_2(t) = c_2e^{2t}$. The phase portrait for this system is given in Figure 1 in Section 1.1 which is reproduced below. And according to the above corollary, the general solution to the original linear system of this example is given by

$$\mathbf{x}(t) = P \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} P^{-1}\mathbf{c}$$

where $\mathbf{c} = \mathbf{x}(0)$, or equivalently by

$$\begin{aligned} x_1(t) &= c_1e^{-t} + c_2(e^{-t} - e^{2t}) \\ x_2(t) &= c_2e^{2t}. \end{aligned} \tag{4}$$

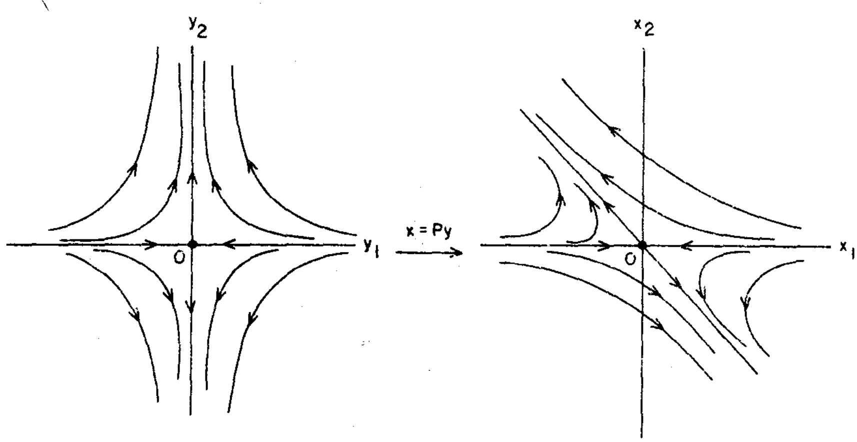

Figure 1 **Figure 2**

The phase portrait for the linear system of this example can be found by sketching the solution curves defined by (4). It is shown in Figure 2. The phase portrait in Figure 2 can also be obtained from the phase portrait in Figure 1 by applying the linear transformation of coordinates $\mathbf{x} = P\mathbf{y}$. Note that the subspaces spanned by the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of the matrix A determine the stable and unstable subspaces of the linear system (1) according to the following definition:

Suppose that the $n \times n$ matrix A has k negative eigenvalues $\lambda_1, \ldots, \lambda_k$ and $n-k$ positive eigenvalues $\lambda_{k+1}, \ldots, \lambda_n$ and that these eigenvalues are distinct. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a corresponding set of eigenvectors. Then the *stable and unstable subspaces of the linear system* (1), E^s *and* E^u, are the linear subspaces spanned by $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ and $\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ respectively; i.e.,

$$E^s = \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$$
$$E^u = \operatorname{Span}\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}.$$

If the matrix A has pure imaginary eigenvalues, then there is also a center subspace E^c; cf. Problem 2(c) in Section 1.1. The stable, unstable and center subspaces are defined for the general case in Section 1.9.

Problem Set 2

1. Find the eigenvalues and eigenvectors of the matrix A and show that $B = P^{-1}AP$ is a diagonal matrix. Solve the linear system $\dot{\mathbf{y}} = B\mathbf{y}$ and then solve $\dot{\mathbf{x}} = A\mathbf{x}$ using the above corollary. And then sketch the phase portraits in both the $\mathbf{x}$ plane and $\mathbf{y}$ plane.

   (a) $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$

   (c) $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$.

2. Find the eigenvalues and eigenvectors for the matrix A, solve the linear system $\dot{\mathbf{x}} = A\mathbf{x}$, determine the stable and unstable subspaces for the linear system, and sketch the phase portrait for

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix} \mathbf{x}.$$

3. Write the following linear differential equations with constant coefficients in the form of the linear system (1) and solve: