MATHEMATICS
FOR THE PHYSICAL SCIENCES

By Herbert S. Wilf

Mathematics for the Physical Sciences

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John Wiley and Sons, Inc. New York . London

Library of Congress Catalog Card Number: 62-15193
Printed in the United States of America

Preface

This book is based on a two-semester course in "The Mathematical Methods of Physics" which I have given in the mathematics department of the University of Illinois in recent years. The audience has consisted primarily of physicists, engineers, and other natural scientists in their first or second year of graduate study. Knowledge of the theory of functions of real and complex variables is assumed

The subject matter has been shaped by the needs of the students and by my own experience. In many cases students who do not major in mathematics have room in their schedules for only one or two mathematics courses. The purpose of this book therefore, is to provide the student with some heavy artillery in several fields of mathematics, in confidence that targets for these weapons will be amply provided by the student's own special field of interest. Naturally, in such an attempt, something must be sacrificed, and I have regarded as most expendable discussions of physical applications of the material being presented

Again, in the short space allotted to each subject there is little chance to develop the theory beyond fundamentals. Thus in each chapter I have gone straight to (what I regard as) the heart of the matter, developing a subject just far enough so that applications can easily be made by the student himself. The exercises at the end of each chapter, along with their solutions at the back of the book, afford some further opportunities for using the theoretical apparatus.

The material herein is, for the most part, classical. The bibliographical references, particularly to journal articles, are given not so much to provide a jumping-off point for further research as to give the reader a feeling for the chronological development of these subjects and for the names of the men who created them.

Finally, I have, where possible, tried to say something about numerical methods for computing the solutions of various kinds of problems. These discussions, while brief, are oriented toward electronic computers and are intended to help bridge the gap between the "there exists" of a pure mathematician and the "find it to three decimal places" of an engineer.

I am indebted to Professor L. A. Rubel for permission to publish Theorem 7 of Chapter 3 here for the first time and to Professor R. P. Jerrard for some of the exercises in Chapter 7. To the well-known volume of Courant and Hilbert I owe the intriguing notion that, even in an age of specialization, it may be possible for physicists and mathematicians to understand each other.

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Philadelphia, Pennsylvania
March, 1962

Contents

chapter 1

Vector spaces and matrices

1.1 VECTOR SPACES

A vector space V is a collection of objects $\mathbf{x}, \mathbf{y}, \ldots$ called vectors, satisfying the following postulates:

(I) If $\mathbf{x}$ and $\mathbf{y}$ are vectors, there is a unique vector $\mathbf{x} + \mathbf{y}$ in V called the sum of $\mathbf{x}$ and $\mathbf{y}$.

(II) If $\mathbf{x}$ is a vector and α a complex number, there is a uniquely defined vector $\alpha\mathbf{x}$ in V satisfying

(1) $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$ (2) $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$

(3) $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$ (4) $1 \cdot \mathbf{x} = \mathbf{x}$

(5) $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ (6) $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$

(III) There is a vector $\mathbf{0}$ in V satisfying

(7) $$\mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$$

for every $\mathbf{x}$ in V, and, further, for every $\mathbf{x}$ in V there is a vector $-\mathbf{x}$ such that

(8) $$\mathbf{x} + (-\mathbf{x}) = \mathbf{0}.$$

We will use the notation $\mathbf{x} - \mathbf{y}$ to mean $\mathbf{x} + (-\mathbf{y})$, as might be expected.

(IV) If $\mathbf{x}$ and $\mathbf{y}$ are vectors in V, there is a uniquely defined complex number $(\mathbf{x}, \mathbf{y})$ called the "inner product" of $\mathbf{x}$ and $\mathbf{y}$ which satisfies

(9) $(\mathbf{x}, \mathbf{y}) = \overline{(\mathbf{y}, \mathbf{x})}$ (10) $(\alpha\mathbf{x}, \mathbf{y}) = \bar{\alpha}(\mathbf{x}, \mathbf{y})$

(11) $(\mathbf{x}, \mathbf{x}) \geq 0$ (12) $(\mathbf{x} + \mathbf{y}, \mathbf{z}) = (\mathbf{x}, \mathbf{z}) + (\mathbf{y}, \mathbf{z})$

(13) $(\mathbf{x}, \mathbf{y} + \mathbf{z}) = (\mathbf{x}, \mathbf{y}) + (\mathbf{x}, \mathbf{z})$ (14) $(\mathbf{x}, \mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{0}$

We state at once that it is not our intention to develop here a purely axiomatic theory of vector spaces. However, in the remainder of this book we shall meet several vector spaces of different types, some of which will not "look like" vector spaces at all. It is most important to note that the only qualifications a system needs in order to be a vector space† are those just set forth, for only in this way can the true unity of such apparently diverse topics as finite dimensional matrices, Fourier series, orthogonal polynomials, integral equations, differential eigenvalue problems, and so on, be perceived. An enlightening exercise for the reader, for example, will be found in analyzing various results as they are proved for special systems, and asking whether or not the properties of the special system were used, or whether, as will more often happen, we have proved a general property of vector spaces.

Example 1. The set of ordered n-tuples of complex numbers $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a vector space V_n (Euclidean n-space) if we define for any vectors

$$\mathbf{x} = (\alpha_1, \ldots, \alpha_n), \ \mathbf{y} = (\beta_1, \ldots, \beta_n),$$

$$\mathbf{x} + \mathbf{y} = (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n) \tag{15}$$

$$\gamma\mathbf{x} = (\gamma\alpha_1, \gamma\alpha_2, \ldots, \gamma\alpha_n) \tag{16}$$

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} \bar{\alpha}_i\beta_i. \tag{17}$$

The complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ are called the *components* of the vector $\mathbf{x}$, and postulates (I)–(IV) are easily verified here by direct calculation. For example, to prove (11),

$$(\mathbf{x}, \mathbf{x}) = \sum_{i=1}^{n} \bar{\alpha}_i\alpha_i = \sum_{i=1}^{n} |\alpha_i|^2 \geq 0. \tag{18}$$

Example 2. The class of functions $f(x)$ of a real variable x, on the interval $[a, b]$ of the real axis for which

$$\int_a^b |f(x)|^2 \, dx < \infty \tag{19}$$

forms a vector space, each vector now being a function satisfying (19). Here the sum of two vectors $f(x)$, $g(x)$ is the vector $f(x) + g(x)$, and the inner product is defined by

$$(f, g) = \int_a^b \overline{f(x)}g(x) \, dx \tag{20}$$

for which the postulates can again easily be shown to be satisfied. This is the space $\mathscr{L}^2(a, b)$, which is of particular interest, for example, in quantum mechanics.

† Our terminology is not conventional. Actually axioms I–III define a vector space, whereas, with axiom IV the structure is sometimes called a "unitary space." We use the term "vector space" for simplicity.

Example 3. Let $w(x) \not\equiv 0$ be a fixed, real-valued, integrable function, defined and non-negative on the interval $[a, b]$ of the real axis. Consider the set of all polynomials

$$f(x) = a_0 + a_1x + \cdots + a_nx^n \tag{21}$$

with real coefficients, and of degree $\leq n$, for some fixed n. This class forms a vector space if addition of two vectors (polynomials) is defined in the obvious way, and if the inner product is given by

$$(f, g) = \int_a^b f(x)g(x)w(x)\,dx. \tag{22}$$

It is in this vector space that we will develop the theory of orthogonal polynomials in the next chapter.

1.2 SCHWARZ'S INEQUALITY AND ORTHOGONAL SETS

Theorem 1. (*Schwarz's inequality*). *Let* **x**, **y** *be vectors in a vector space V. Then*

$$|(\mathbf{x}, \mathbf{y})|^2 \leqq (\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y}) \tag{23}$$

the sign of equality holding if and only if there is a complex number α *such that* $\mathbf{x} = \alpha\mathbf{y}$ (i.e., *if* **x** *and* **y** *are proportional*).

Proof. Let λ be any real number. By (11),

$$(\mathbf{x} + \lambda(\mathbf{y}, \mathbf{x})\mathbf{y}, \quad \mathbf{x} + \lambda(\mathbf{y}, \mathbf{x})\mathbf{y}) \geqq 0. \tag{24}$$

Hence by (9), (12), and (13),

$$0 \leqq (\mathbf{x}, \mathbf{x}) + 2\lambda\,|(\mathbf{x}, \mathbf{y})|^2 + \lambda^2\,|(\mathbf{x}, \mathbf{y})|^2\,(\mathbf{y}, \mathbf{y}) \tag{25}$$

for all real λ.

Thus the discriminant of this quadratic polynomial is not positive, that is,

$$|(\mathbf{x}, \mathbf{y})|^4 - (\mathbf{x}, \mathbf{x})\,|(\mathbf{x}, \mathbf{y})|^2\,(\mathbf{y}, \mathbf{y}) \leqq 0.$$

If $(\mathbf{x}, \mathbf{y}) \neq 0$, we get

$$|(\mathbf{x}, \mathbf{y})|^2 \leqq (\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y}) \tag{26}$$

whereas if $(\mathbf{x}, \mathbf{y}) = 0$, (26) is obvious. Finally, suppose the sign of equality holds in (26). Then in (25) we have a quadratic polynomial with zero discriminant, which therefore is zero for some real value of λ, say λ_0. Referring to (14) and (24) we see that

$$\mathbf{x} + \lambda_0(\mathbf{y}, \mathbf{x})\mathbf{y} = 0 \tag{27}$$

which is to say that x is proportional to y. Conversely, if $\mathbf{x} = \beta\mathbf{y}$, substitution in (23) shows at once that the sign of equality holds.

Two vectors **x** and **y** are said to be *orthogonal* if

$$(\mathbf{x}, \mathbf{y}) = 0. \tag{28}$$

The *length* of a vector **x** is defined by

$$\|\mathbf{x}\| = (\mathbf{x}, \mathbf{x})^{1/2} \tag{29}$$

and is always a non-negative real number. In terms of the length, Schwarz's inequality (23) reads

$$|(\mathbf{x}, \mathbf{y})| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|. \tag{30}$$

A finite or infinite sequence of vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ is called an *orthogonal set* if

$$(\mathbf{x}_i, \mathbf{x}_j) = 0 \qquad (i \neq j;\ i, j = 1, 2, 3, \ldots) \tag{31}$$

and an *orthonormal* set if, in addition to (31), we have also

$$\|\mathbf{x}_i\| = 1 \qquad (i = 1, 2, \ldots). \tag{32}$$

The two conditions (31) and (32) are frequently combined in the form

$$(\mathbf{x}_i, \mathbf{x}_j) = \delta_{ij} \qquad (i, j = 1, 2, \ldots) \tag{33}$$

where δ_{ij}, the "Kronecker delta," is defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{34}$$

A vector **x** of length unity is said to be *normalized.*

Now let **f** be an arbitrary vector in a vector space V, and let† $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be an orthonormal set in V. The numbers

$$\gamma_\nu = (\mathbf{x}_\nu, \mathbf{f}) \qquad (\nu = 1, 2, \ldots) \tag{35}$$

are called the *Fourier coefficients* of **f** with respect to the set $\mathbf{x}_1, \mathbf{x}_2, \ldots$. These coefficients are of considerable importance in applications. As an example, consider the following approximation problem: let n be a fixed integer, **f** a given vector of a vector space V, and $\mathbf{x}_1, \ldots, \mathbf{x}_n$ an orthonormal set lying in V. It is required to find numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ for which the vector

$$\mathbf{h} = \alpha_1 \mathbf{x}_1 + \cdots + \alpha_n \mathbf{x}_n \tag{36}$$

is the best possible approximation to **f** in the sense that $\|\mathbf{f} - \mathbf{h}\|$ is as small as possible.

† For existence see § 1.3.

To solve this problem, we have

$$
\begin{aligned}
(37)\quad \|\mathbf{f}-\mathbf{h}\|^2 &= (\mathbf{f}-\mathbf{h}, \mathbf{f}-\mathbf{h}) \\
&= (\mathbf{f}-\alpha_1\mathbf{x}_1-\cdots-\alpha_n\mathbf{x}_n, \mathbf{f}-\alpha_1\mathbf{x}_1-\cdots-\alpha_n\mathbf{x}_n) \\
&= (\mathbf{f},\mathbf{f})-\alpha_1(\mathbf{f},\mathbf{x}_1)-\cdots-\alpha_n(\mathbf{f},\mathbf{x}_n) \\
&\quad -\bar{\alpha}_1(\mathbf{x}_1,\mathbf{f})-\cdots-\bar{\alpha}_n(\mathbf{x}_n,\mathbf{f})+|\alpha_1|^2+\cdots+|\alpha_n|^2. \\
&= (\mathbf{f},\mathbf{f})-\alpha_1\bar{\gamma}_1-\bar{\alpha}_1\gamma_1-\cdots-\alpha_n\bar{\gamma}_n-\bar{\alpha}_n\gamma_n \\
&\quad +|\alpha_1|^2+\cdots+|\alpha_n|^2 \\
&= (\mathbf{f},\mathbf{f})+(|\alpha_1|^2-2\mathrm{Re}\,\alpha_1\bar{\gamma}_1)+\cdots+(|\alpha_n|^2-2\mathrm{Re}\,\alpha_n\bar{\gamma}_n) \\
&= (\mathbf{f},\mathbf{f})+|\alpha_1-\gamma_1|^2+\cdots+|\alpha_n-\gamma_n|^2-|\gamma_1|^2-\cdots-|\gamma_n|^2.
\end{aligned}
$$

Now, remembering that $\mathbf{f}, \gamma_1, \ldots, \gamma_n$ are fixed, and only $\alpha_1, \ldots, \alpha_n$ are at our disposal, it is plain that the choice of $\alpha_1, \ldots, \alpha_n$ which minimizes the "least squares error" $\|\mathbf{f}-\mathbf{h}\|^2$ is

$$(38)\qquad \alpha_\nu = \gamma_\nu = (\mathbf{x}_\nu, \mathbf{f}) \qquad (\nu = 1, 2, \ldots, n).$$

Furthermore, if we make this optimal choice of the α_ν as the Fourier coefficients of $\mathbf{f}$, (37) shows clearly that

$$0 \leqq (\mathbf{f},\mathbf{f}) - |\gamma_1|^2 - \cdots - |\gamma_n|^2$$

or

$$(39)\qquad \sum_{\nu=1}^{n} |\gamma_\nu|^2 \leqq (\mathbf{f},\mathbf{f}).$$

This inequality, known as Bessel's inequality, is seen to be a property of the vector $\mathbf{f}$ and the set $\mathbf{x}_1, \ldots, \mathbf{x}_n$ only, and therefore expresses a general property of Fourier coefficients.

It may happen that a given orthonormal set $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ has the property that every vector $\mathbf{f}$ in the space V can be approximated arbitrarily closely by taking n, the number of vectors used from the set, large enough.

More precisely, let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be an orthonormal set with the property that if $\varepsilon > 0$ and an arbitrary vector $\mathbf{f}$ of V are given, there is an n for which the vector (36) with (38) implies

$$\|\mathbf{f}-\mathbf{h}\| < \varepsilon.$$

We then say that $\mathbf{x}_1, \mathbf{x}_2, \ldots$ is a *complete* orthonormal set. The following theorems are now clear:

Theorem 2. *Let $\mathbf{x}_1, \mathbf{x}_2, \ldots$ be a complete orthonormal set in a vector space V and let $\mathbf{f}$ be a vector of V. Then*

$$(40)\qquad \sum_{\nu=1}^{\infty} |(\mathbf{x}_\nu, \mathbf{f})|^2 = (\mathbf{f},\mathbf{f}) \qquad \textit{(Parseval's identity)}.$$

Theorem 3. *(The Riemann-Lebesgue Lemma). If* $\mathbf{x}_1, \mathbf{x}_2, \ldots$ *is an infinite orthonormal set and* $\mathbf{f}$ *is any vector of* V, *then*

$$|(\mathbf{x}_\nu, \mathbf{f})| \to 0 \qquad (\nu \to \infty). \tag{41}$$

Since the series on the left side of (39) obviously converges, its terms must approach zero.

1.3 LINEAR DEPENDENCE AND INDEPENDENCE

The vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are said to be *linearly dependent* if there are constants $\alpha_1, \ldots, \alpha_n$ not all zero, such that

$$\alpha_1\mathbf{x}_1 + \cdots + \alpha_n\mathbf{x}_n = \mathbf{0}. \tag{42}$$

Otherwise the vectors are *linearly independent.*

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be linearly independent. We wish to transform the set $\mathbf{x}_1, \ldots, \mathbf{x}_n$ into a new set $\mathbf{y}_1, \ldots, \mathbf{y}_n$ having the properties: (i) $\mathbf{y}_1, \ldots, \mathbf{y}_n$ is an orthonormal set, (ii) each $\mathbf{y}_i$ is a linear combination of the $\mathbf{x}_j$ $(j = 1, \ldots, n)$. This may be accomplished by the following procedure, called the ***Gram-Schmidt process.***

First, take

$$\mathbf{y}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|}. \tag{43}$$

Then, clearly $\|\mathbf{y}_1\| = 1$. Next, assume

$$\mathbf{y}_2' = \mathbf{x}_2 - \lambda_1\mathbf{y}_1$$

and determine the constant λ_1, such that $(\mathbf{y}_2', \mathbf{y}_1) = 0$, i.e., take

$$\lambda_1 = (\mathbf{y}_1, \mathbf{x}_2).$$

Since $\mathbf{x}_1, \mathbf{x}_2$ are linearly independent, $\mathbf{y}_2' \neq \mathbf{0}$, and we set

$$\mathbf{y}_2 = \frac{\mathbf{y}_2'}{\|\mathbf{y}_2'\|}$$

In general, if $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k$ have been constructed, write

$$\mathbf{y}_{k+1}' = \mathbf{x}_{k+1} - \sigma_1\mathbf{y}_1 - \cdots - \sigma_k\mathbf{y}_k \tag{44}$$

and determine the constants $\sigma_1, \ldots, \sigma_k$ so that

$$(\mathbf{y}_{k+1}', \mathbf{y}_j) = 0 \qquad (j = 1, 2, \ldots, k) \tag{45}$$

that is, choose

$$\sigma_j = (\mathbf{y}_j, \mathbf{x}_{k+1}) \qquad (j = 1, 2, \ldots, k). \tag{46}$$

As before, $\mathbf{y}'_{k+1} \neq \mathbf{0}$, and taking $\mathbf{y}_{k+1} = \mathbf{y}'_{k+1}/\|\mathbf{y}'_{k+1}\|$, we have constructed the next vector in the set.

A vector space V is said to be of *dimension n* if it contains n linearly independent vectors, but every $n + 1$ vectors are linearly dependent. A space which for every integer n contains n linearly independent vectors is said to be *infinite dimensional*. By virtue of the Gram-Schmidt process we see that the dimension of a vector space is also the length of the longest orthonormal set contained in the space.

A set of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is said to *span* a vector space V if every vector of V is a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, that is, if $\mathbf{f}$ is an arbitrary vector of V, there exist complex numbers $\alpha_1, \alpha_2, \ldots$ such that

$$\mathbf{f} = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \cdots. \tag{47}$$

A set of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots$ is said to form a *basis* for a vector space V if (i) the set spans the space and (ii) the set is linearly independent.

1.4 LINEAR OPERATORS ON A VECTOR SPACE

A linear operator on a vector space V is a rule which assigns to each vector $\mathbf{f}$ of V a unique vector $T\mathbf{f}$ of V, in such a way that

$$T(\alpha\mathbf{f} + \mathbf{g}) = \alpha T\mathbf{f} + T\mathbf{g} \tag{48}$$

for every pair of vectors $\mathbf{f}$, $\mathbf{g}$ in V and every complex number α.

Example 1. For Euclidean n-space, the operator which associates with

$$\mathbf{x} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

the vector

$$T\mathbf{x} = (\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \cdots, \alpha_1 + \alpha_2 + \cdots + \alpha_n)$$

is a linear operator.

Example 2. In $\mathscr{L}^2(a, b)$ the rule which associates with the vector $f(x)$ the vector

$$Tf(x) = \int_a^x f(y)\,dy \qquad (a \leq x \leq b)$$

is a linear operator.

Henceforth the term "operator" will invariably refer to a linear operator on the space in question.

The *identity operator I* is the operator which assigns to any vector $\mathbf{f}$ the vector $\mathbf{f}$ itself, i.e.,

$$I\mathbf{f} = \mathbf{f} \qquad (\text{all } \mathbf{f} \text{ in } V). \tag{49}$$

This is clearly linear. Two operators T, U are said to be *equal* if their effect on every vector of V is the same, that is, $T = U$ means

$$TF = UF \qquad (\text{all } \mathbf{f} \text{ in } V). \tag{50}$$

The product TU of two operators T and U is defined by

$$(TU)\mathbf{f} = T(U\mathbf{f}). \tag{51}$$

In general, we do not have $TU = UT$. If $TU = UT$, however, we say that T *commutes* with U, and, in any case, the commutator $[T, U]$ of two operators is

$$[T, U] = TU - UT \tag{52}$$

so that two operators commute if and only if their commutator is the zero operator.

Let T be an operator on V. There may or may not be an operator U on V such that

$$UT = TU = I.$$

If there is such a U, we say that U is the *inverse* of T, and write $U = T^{-1}$. Hence

$$T^{-1}T = TT^{-1} = I. \tag{53}$$

The operator T^{-1}, when it exists, "undoes" the work of T in the sense that if $\mathbf{f}$ is any vector of V, we have

$$T^{-1}(T\mathbf{f}) = (T^{-1}T)\mathbf{f} = I\mathbf{f} = \mathbf{f}. \tag{54}$$

An operator which has an inverse will be called *nonsingular*, otherwise the operator is *singular*. A simple property of the inverse operator is

Theorem 4. *The inverse of a product is the product of the inverses in reverse order*, i.e.,

$$(ST)^{-1} = T^{-1}S^{-1} \tag{55}$$

if S and T are nonsingular,

Proof.

$$(ST)(T^{-1}S^{-1}) = S(TT^{-1})S^{-1} = SIS^{-1} = SS^{-1} = I$$
$$(T^{-1}S^{-1})(ST) = T^{-1}(S^{-1}S)T = T^{-1}IT = T^{-1}T = I$$

which was to be shown.

1.5 EIGENVALUES AND HERMITIAN OPERATORS

Let T be an operator on a vector space V. Among all the vectors of V, there may be some nonzero vectors which, when operated on by T, do not