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Edwin E. Moise

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Edwin E. Moise
Department of Mathematics
Queens College, CUNY
Flushing, N.Y. 11367

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Preface

Geometric topology may roughly be described as the branch of the topology of manifolds which deals with questions of the existence of homeomorphisms. Only in fairly recent years has this sort of topology achieved a sufficiently high development to be given a name, but its beginnings are easy to identify. The first classic result was the Schönflies theorem (1910), which asserts that every 1-sphere in the plane is the boundary of a 2-cell.

In the next few decades, the most notable affirmative results were the "Schönflies theorem" for polyhedral 2-spheres in space, proved by J. W. Alexander [A_1], and the triangulation theorem for 2-manifolds, proved by T. Radó [R_1]. But the most striking results of the 1920s were negative. In 1921 Louis Antoine [A_4] published an extraordinary paper in which he showed that a variety of plausible conjectures in the topology of 3-space were false. Thus, a (topological) Cantor set in 3-space need not have a simply connected complement; therefore a Cantor set can be imbedded in 3-space in at least two essentially different ways; a topological 2-sphere in 3-space need not be the boundary of a 3-cell; given two disjoint 2-spheres in 3-space, there is not necessarily any third 2-sphere which separates them from one another in 3-space; and so on and on. The well-known "horned sphere" of Alexander [A_2] appeared soon thereafter. Much later, in 1948, these results were extended and refined (and in some cases redone) by Ralph H. Fox and Emil Artin [FA].

The affirmative theory was resumed with the author's proof [M_1]–[M_5] that every 3-manifold can be triangulated, and that every two triangulations of the same 3-manifold are combinatorially equivalent. The second of these statements is the *Hauptvermutung* of Steinitz. Then, in 1957, C. D. Papakyriakopoulos revolutionized the field by proving the Loop theorem.

A *loop* is a mapping of a 1-sphere into a space. The Loop theorem is as follows. Let M be a polyhedral 3-manifold with boundary, and let B be its boundary. Let L be a loop in B, and suppose that L is contractible in M but not in B. Then there is a polyhedral 2-cell D in M, with its boundary in B, such that the boundary of D is not contractible in B.

In 1971 Peter B. Shalen [S_1] found a new proof of the triangulation theorem and *Hauptvermutung*. His proof is "almost PL," in the sense that the set-theoretic part of the argument is elementary, almost to the point of triviality, and the main substance of the proof belongs to piecewise linear topology, with heavy use of the Loop theorem. Following Shalen's example, and using some of his methods, especially at the beginning, the author developed the proofs presented below, in Sections 30–36.

The historical account just given will also serve as a summary of the contents of this book. The treatment of plane topology is rudimentary. Here traditional material has been reformulated, in "almost PL" terms, in the hope that this will help, as an introduction to the methods to be used in three dimensions, and that it will bring three-dimensional ideas into sharper focus. The proofs of the triangulation theorem and *Haupvermutung* are largely new, as explained above. So also is our proof of the Schönflies theorem. But most of the time, we have followed the historical order. This is not because we were trying to write a history; far from it. The point, rather, is that the historical order was the natural order of intellectual motivation.

Recently, A. J. S. Hamilton [H_3] has published yet another proof of the triangulation theorem, based on methods which had been developed by Kirby and Siebenmann for use in higher dimensions. His proof and presentation are shorter and more learned than ours, by a very wide margin in each respect.

This is a textbook and not a treatise, and the difference is important. A presentation which looks elegant to a professional expert may not seem elegant, or even intelligible, to a student who is encountering certain ideas for the first time. We have furnished a very large number of problems. One way to teach a course based on this book is to spend most of the classroom time on discussion of problems, treating much of the text as outside reading. A warning is needed about the style in which the problems are written. This warning is given at the end of the preface, in the hope of minimizing the chance that it will be overlooked.

References to the literature, in this book, are meager by normal standards. Whenever I was indebted to a particular author, and knew it, I have given a reference. But I have made no systematic effort to search the literature thoroughly enough to find out who deserves credit for what. Many of the proofs below are new, and many others must be adaptations (conscious or not) of folklore. Here again I have made no attempt to find out which is which. I believe, however, that all papers published since 1945 have been cited when they should have been.

In 1975–76 at the University of Texas, and earlier at the University of Wisconsin, the manuscript of this book was used in seminars conducted by Prof. R. H. Bing. The faculty members participating included Profs. Bing, Bruce Palka, Carl Pixley, Michael Starbird, and Gerard Venema. The students included Ms. Mary Parker, Ms. Fay Shaparenko, and Messrs. William E. Bell, Joseph M. Carter, Lee Leonard, Wayne Lewis, Gary Richter, and Frank Shirley. I received long critical reports prepared by Messrs. Bell, Henderson, and Richter. If I had not had the benefit of these reports, then the text below would include more errors and obscurities than it does now. Finally, thanks are due to Mr. Michael Weinstein, who edited the manuscript for Springer-Verlag. In the course of dealing with matters of form, Mr. Weinstein detected a dismaying number of minor lapses which the rest of us had missed. The responsibility for the remaining defects is of course my own.

Finally, a word of warning about the problems in this book. These are composed in a way which may not be familiar. Most of them state true theorems, extending or elucidating the preceding section of the text. But in a very large number of them, false propositions are stated as if they were true. Here it is the student's job to discover that they are false, and find counter-examples. Problems cannot be relied on to appear in the approximate order of their difficulty. Some of them turn out, on examination, to be trivial, but some are very difficult. Thus the problems are intended to furnish the student with an opportunity to work on mathematics under conditions which are not hopelessly remote from real life.

Edwin E. Moise

New York City
January, 1977

Contents

Introduction 0

We shall use the following definitions, notations, and conventions, most of them standard, but a few not.

$\mathbf{R}$ is the set of all real numbers. $\mathbf{R}^+$ is the set of all nonnegative real numbers. $\mathbf{Z}$ is the set of all integers. $\mathbf{Z}^+$ is the set of all nonnegative integers. $\mathbf{R}^n$ is Cartesian n-space, with the usual linear structure, the usual distance function, and the usual topology. (We shall always be dealing with cases in which $n \leqslant 3$.) The empty set is denoted by $\varnothing$.

A *metric space* is a pair $[X, d]$, where X is a nonempty set and d is a function $X \times X \to \mathbf{R}$, subject to the usual conditions:

(D.1) $d(P, Q) \geqslant 0$ always.
(D.2) $d(P, Q) = 0$ if and only if $P = Q$.
(D.3) $d(P, Q) = d(Q, P)$ always.
(D.4) (the triangular property) $d(P, Q) + d(Q, R) \geqslant d(P, R)$ always.

Under these conditions, d is called a *distance function* for X. By abuse of language, we may refer to the set X as a metric space, if it is clear what distance function is meant.

In a metric space $[X, d]$, for each P in X and each $\varepsilon > 0$, we define the (open) ε-neighborhood of P as the set

$$N(P, \varepsilon) = \{Q \mid Q \in X \text{ and } d(P, Q) < \varepsilon\}.$$

More generally, for each $M \subset X$, and each $\varepsilon > 0$, the ε-neighborhood of M is

$$N(M, \varepsilon) = \{Q \mid Q \in X \quad \text{and} \quad d(P, Q) < \varepsilon \text{ for some } P \in M\}.$$

We define

$$\mathfrak{N} = \mathfrak{N}(d) = \{N(P, \varepsilon) \mid P \in X \quad \text{and} \quad \varepsilon > 0\}.$$

$\mathfrak{N}(d)$ is called the *neighborhood system induced by d*. A set $U \subset X$ is *open* if it is the union of a collection of elements of $\mathfrak{N}$. The set of all open sets is $\mathcal{O} = \mathcal{O}(\mathfrak{N}) = \mathcal{O}(\mathfrak{N}(d))$. $\mathcal{O}$ is called the *topology induced by* $\mathfrak{N}$ (or by d). Under these conditions, the pair $[X, \mathcal{O}]$ is a topological space, in the usual sense; that is:

(O.1) $\varnothing \in \mathcal{O}$.
(O.2) $X \in \mathcal{O}$.
(O.3) $\mathcal{O}$ contains every union of elements of $\mathcal{O}$.
(O.4) $\mathcal{O}$ contains every finite intersection of elements of $\mathcal{O}$.

Closed sets, limit points, and the closure $\overline{M}$ of a set $M \subset X$ are defined as usual. The closure may also be denoted by Cl M.

In a topological space, let M and N be sets such that N contains an open set which contains M. Then N *is a neighborhood of* M. (Note that this is not a new definition of the term *neighborhood*; rather, it is a definition of the relation *is a neighborhood of*.)

Let $[X, \mathcal{O}]$ be a topological space. For each nonempty set $M \subset X$, let

$$\mathcal{O}|M = \{M \cap U | U \in \mathcal{O}\}.$$

Then $\mathcal{O}|M$ is called the *subspace topology for* M, and the pair $[M, \mathcal{O}|M]$ is called a *subspace* of $[X, \mathcal{O}]$. In this book, when subsets of topological spaces are regarded as spaces in themselves, the subspace topology will always be intended.

Let $\mathbf{V}$ be a subset of $\mathbf{R}^m$, such that $\mathbf{V}$ forms a vector space relative to the operations already defined in $\mathbf{R}^m$. Let $v_0 \in \mathbf{R}^m$, and let

$$H = \mathbf{V} + v_0 = \{w | w = v + v_0 \text{ for some } v \in \mathbf{V}\}.$$

Then H is a *hyperplane*. If dim $\mathbf{V} = k$, then H is a *k-dimensional* hyperplane. If $V \subset \mathbf{R}^m$, and no k-dimensional hyperplane, with $k < m$, contains more than $k + 1$ of the points of V, then V is *in general position* in $\mathbf{R}^m$.

A set $W \subset \mathbf{R}^m$ is *convex* if for each $v, w \in W$, W contains the segment

$$vw = \{\alpha v + \beta w | \alpha, \beta \geqslant 0, \alpha + \beta = 1\}.$$

The *convex hull* of a set $X \subset \mathbf{R}^m$ is the smallest convex subset of $\mathbf{R}^m$ that contains X (that is, the intersection of all convex subsets of $\mathbf{R}^m$ that contain X).

Let $V = \{v_0, v_1, \ldots, v_n\}$ be a set of $n + 1$ points, in general position in $\mathbf{R}^m$, with $n \leqslant m$. Then the *n-dimensional simplex* (or *n*-simplex)

$$\sigma^n = v_0 v_1 \ldots v_n$$

is the convex hull of V. The points of V are *vertices* of σ^n. The convex hull τ of a nonempty subset W of V is called a *face* of σ^n. If τ is a k-simplex, then τ is called a *k-face* of σ^n. (A 1-simplex is called an *edge*.) Under these conditions, we write $\tau < \sigma^n$. (This allows the case $\tau = \sigma^n$.) A (*Euclidean*) *complex* is a collection K of simplexes in a space $\mathbf{R}^m$, such that

(K.1) K contains all faces of all elements of K.
(K.2) If $\sigma, \tau \in K$, and $\sigma \cap \tau \neq \varnothing$, then $\sigma \cap \tau$ is a face both of σ and of τ.
(K.3) Every σ in K lies in an open set U which intersects only a finite number of elements of K.

The vertices of the elements of K will be called vertices of K. For each $i \geqslant 0$, K^i is the *i-skeleton* of K, that is, the set of all simplexes of K that have dimension $\leqslant i$.

These definitions will of course be generalized later, but for quite a while we shall be concerned only with finite complexes in $\mathbf{R}^2$.

If K is a complex, then $|K|$ denotes the union of the elements of K, with the subspace topology induced by the topology of $\mathbf{R}^m$. (Thus we shall think of $|K|$ ambiguously, as either a set or a space.) Such a set is called a *polyhedron*. If K is a finite complex, then $|K|$ is a *finite polyhedron*.

The word *function* will be used in its most general sense. Thus a function

$$f\colon A \to B$$

is a triplet $[f, A, B]$, where A and B are nonempty sets, and f is a collection of ordered pairs (a, b), with $a \in A$, such that (1) each $a \in A$ is the first term of exactly one pair in f, and (2) the second term of a pair in f is always an element of B. We define $f(a)$ $(a \in A)$ and $f(A')$ $(A' \subset A)$ as usual; and we define

$$f^{-1}(b) = \{a \mid f(a) = b\} \qquad (b \in B),$$

$$f^{-1}(B') = \{a \mid f(a) \in B'\} \qquad (B' \subset B).$$

If $f(a) = f(a') \Rightarrow a = a'$, then f is *injective*. If $f(A) = B$, then f is *surjective*, and we write

$$f\colon A \twoheadrightarrow B.$$

If both these conditions hold, then f is *bijective*, and we write

$$f\colon A \leftrightarrow B.$$

A is called the *domain*, and B the *codomain*. (Note that the term *surjective* would have no meaning if the codomain were not regarded as part of the definition of the function.)

Barycentric coordinates, for a (Euclidean) simplex σ^n, are defined as usual. (See Problems 0.10–0.15.) The barycentric coordinates of the points P of σ^n are linear functions of the Cartesian coordinates, and vice versa. A function $f\colon \sigma \to \tau$ is *linear* if the coordinates of a point $f(P)$ are linear functions of those of P (in either sense of the word *coordinate*). If also vertices are mapped onto vertices, then f is *simplicial*.

Let G and H be collections of sets. If every element of G is a subset of some element of H, then G is a *refinement* of H, and we write $G \leqslant H$.

Let K and L be complexes, in the same space $\mathbf{R}^n$. If $L \leqslant K$, and $|L| = |K|$, then L is a *subdivision* of K, and we write $L < K$.

Theorem 1. *Every two subdivisions of the same complex have a common subdivision.*

Let $[X, \mathcal{O}]$ and $[Y, \mathcal{O}']$ be topological spaces, and let $f: X \to Y$ be a function. If for each open set U in Y, $f^{-1}(U)$ is open in X, then f is a *continuous function*, or a *mapping*. If such an f is bijective, and both f and f^{-1} are mappings, then f is a *homeomorphism*. If there is a homeomorphism $f: X \leftrightarrow Y$, then the spaces are *homeomorphic*.

Let K and L be complexes, and let f be a mapping $|K| \to |L|$. If each mapping $f|\sigma$ $(\sigma \in K)$ is simplicial, then f is *simplicial*. If there is a subdivision K' of K such that each mapping $f|\sigma$ $(\sigma \in K')$ maps σ linearly into a simplex of L, then f is *piecewise linear*. Hereafter, PL stands for *piecewise linear*, and a PLH is a *piecewise linear homeomorphism*.

Let K and L be complexes, let ϕ be a bijection $K^0 \leftrightarrow L^0$, and for each $v \in K^0$, let $v' = \phi(v)$. Suppose that if $v_0v_1 \ldots v_n \in K$, then $v_0'v_1' \ldots v_n' \in L$, and conversely. Then ϕ is an *isomorphism* between K and L. If there is such a ϕ, then K and L are *isomorphic*. If K and L are complexes, and have subdivisions K', L' which are isomorphic, then K and L are *combinatorially equivalent*, and we write

$$K \sim_c L.$$

Theorem 2. $K \sim_c L$ *if and only if* $|K|$ *is the image of* $|L|$ *under a* PLH.

Theorem 3. *Combinatorial equivalence is an equivalence relation.*

PROOF (SKETCH). By Theorem 1, the composition of two piecewise linear homeomorphisms is a PLH. Now use Theorem 2. □

An *n-cell* is a space homeomorphic to an n-simplex. A 1-cell is ordinarily called an *arc*, and a 2-cell is often called a *disk*. A *combinatorial n-cell* is a complex which is combinatorially equivalent to an n-simplex (or, more precisely, to a complex consisting of an n-simplex and its faces).

In a topological space, a set A *is dense in* a set B if $A \subset B \subset \bar{A}$. A topological space $[X, \mathcal{O}]$ (or a metric space $[X, d]$) is *separable* if some countable set is dense in X.

An *n-manifold* is a separable metric space M^n in which every point has a neighborhood homeomorphic to $\mathbf{R}^n$. If every point lies in an open set whose closure is an n-cell, then M^n is an *n-manifold with boundary*. The *interior* $\operatorname{Int} M^n$ of M^n is the set of all points of M^n that have open Euclidean neighborhoods in M^n (that is, neighborhoods homeomorphic to $\mathbf{R}^n$); and the *boundary* $\operatorname{Bd} M^n$ is the set of all points of M^n that do not. Thus an n-manifold with boundary is an n-manifold if and only if $\operatorname{Bd} M^n = \emptyset$.

The manifold-theoretic boundary, as just defined, is in general different from the topological *frontier* of a set U in a space X. This is

$$\operatorname{Fr} U = \operatorname{Fr}_X U = \bar{U} \cap \overline{X - U}.$$

Only in very special cases are these the same. For example, if M^2 is closed in $\mathbf{R}^2$, then it turns out that Bd $M^2 =$ Fr M^2; but if we regard M^2 as a subspace of $\mathbf{R}^3$, then Bd M^2 is the same as before, while Fr M^2 becomes all of M^2. (The proofs are far from trivial.) Similarly, except in very special cases, Int M^n is different from the topological interior of a set M in a space X; the latter is the union of all open sets that lie in M.

Let K be a complex, such that the space $M = |K|$ is an n-manifold (or an n-manifold with boundary). Then K is a *triangulated n-manifold* (or a *triangulated n-manifold with boundary*). Sometimes, by abuse of language, we may apply the latter terms to the space $M = |K|$, if it is clear what triangulation is intended.

In addition to Bd and Fr, we now have yet a third kind of "boundary." Let K be a triangulated n-manifold with boundary. Then the *combinatorial boundary* ∂K of K is the set of all $(n-1)$-simplexes of K that lie in only one n-simplex of K (together with all faces of such $(n-1)$-simplexes). Note that ∂ is an operation on complexes to complexes, and not on spaces to spaces. It is easy to show that $|\partial K|$ is invariant under subdivision of K, and hence that $f(|\partial K|) = \partial f(|K|)$ whenever f is a PLH. Thus ∂ is adequate for the purposes of strictly PL topology, in which combinatorial structures are the sole objects of investigation. But ∂ is not adequate for our present purposes, because we propose to investigate the relation between combinatorial structures and purely topological structures. We shall show (Theorem 4.9) that if K is a triangulated 2-manifold with boundary, then Bd $|K| = |\partial K|$. The proof uses the Jordan curve theorem (Theorem 4.3). The corresponding theorem for 3-manifolds with boundary is of a higher order of difficulty. In Section 23, we shall deduce it from the following classical result of L. E. J. Brouwer.

Theorem 4 (Invariance of domain). *Let U be a subset of $\mathbf{R}^n$, such that U is homeomorphic to $\mathbf{R}^n$. Then U is open.*

See W. Hurewicz and H. Wallman [HW], p. 95.

It may be possible to avoid the use of Brouwer's theorem (or some equally deep result in a continuous homology theory) by a long series of ad hoc devices; but this hardly seems worth the trouble, even if it can be done, and the author does not propose to find out whether it can be done.

In a complex K, for each vertex v, St v is the complex consisting of all simplexes of K that contain v, together with all their faces. This is the *star* of v in K. The *link* $L(v)$ of v in K is the set of all simplexes of St v that do not contain v. If $|K|$ is an n-manifold, and each complex St v is a combinatorial n-cell, then K is a *combinatorial n-manifold*. Similarly for manifolds with boundary.

The above definitions are based, at this stage, on the definition of a (Euclidean) complex. A later generalization of the idea of a complex will give a more general definition of a combinatorial manifold.

We shall assume that the reader knows the bare rudiments of the homology theory of complexes. We shall always use integers as coefficients; thus the n-dimensional homology group $H_n(K)$ will always be the group $H_n(K, \mathbf{Z})$. We shall never use relative homology, singular homology, or cohomology.

PROBLEM SET 0

See the remarks on problems, at the end of the preface. Prove or disprove the following propositions.

1. Let $[X, d]$ be a metric space, let $\mathfrak{N} = \mathfrak{N}(d)$, and let $\mathcal{O} = \mathcal{O}(\mathfrak{N})$. Then $\mathcal{O}$ satisfies Conditions O.1–O.4 of the definition of a topological space.

Definition. Let d and d' be two distance functions for the same nonempty set X. If $\mathcal{O}(\mathfrak{N}(d)) = \mathcal{O}(\mathfrak{N}(d'))$, then d and d' are *equivalent*.

2. Let $[X, d]$ be a metric space. Then there is a bounded distance function d' for X such that d and d' are equivalent.

Definition. A *Hausdorff* space is a topological space in which every two points lie in disjoint open sets.

3. Let $[X, \mathcal{O}]$ be a topological space in which every point has an open neighborhood homeomorphic to $\mathbf{R}^2$. Then $[X, \mathcal{O}]$ is Hausdorff.

4. Let $[X, \mathcal{O}]$ be a topological space; and suppose that for every topological space $[Y, \mathcal{O}']$, every function $f: X \to Y$ is continuous. What can we conclude about $\mathcal{O}$? In particular, does it follow that $[X, \mathcal{O}]$ is metrizable, in the sense that $\mathcal{O} = \mathcal{O}(\mathfrak{N}(d))$ for some distance function d?

5. Let C be a circle in $\mathbf{R}^2$. Then C is in general position in $\mathbf{R}^2$.

6. Let C be a circle in $\mathbf{R}^3$. Then C is in general position in $\mathbf{R}^3$.

7. $\mathbf{R}^3$ contains an infinite set which is in general position in $\mathbf{R}^3$.

8. Let K and L be collections of simplexes in $\mathbf{R}^n$, satisfying K.1 and K.2 in the definition of a complex, but not necessarily K.3. The relation of *isomorphism* between K and L is defined in exactly the same way as for complexes. If there is an isomorphism between K and L, then there is a homeomorphism between $|K|$ and $|L|$. (Here, as for complexes, $|K|$ is the union of the elements of K; similarly for L. $|K|$ and $|L|$ are being regarded as spaces, with the subspace topology.)

9. For each $W \subset \mathbf{R}^m$, the convex hull of W is convex.

10. Let $V = \{v_0, v_1, \ldots, v_n\}$ be in general position in $\mathbf{R}^m$, with $n < m$. Let

$$\tau^n = \left\{ v \mid v = \sum_{i=0}^{n} \alpha_i v_i,\ \alpha_i \geq 0,\ \Sigma \alpha_i = 1 \right\}.$$

Then τ^n is convex.

11. Let τ^n be as in Problem 10, and let $v \in \sigma^n$, with $v \neq v_0$. Let

$$\tau^{n-1} = \left\{ w | w = \sum_{i=1}^{n} \beta_i v_i, \beta_i \geq 0, \Sigma \beta_i = 1 \right\}.$$

Then there is point w of τ^{n-1} such that $v \in v_0 w$.

12. Let V and τ^n be as in Problem 10. Then every convex set that contains V contains τ^n.

13. $\sigma^n = \tau^n$. That is,

$$v_0 v_1 \ldots v_n = \{ v | v = \Sigma \alpha_i v_i, \alpha_i \geq 0, \Sigma \alpha_i = 1 \}.$$

14. Given $V = \{v_0, v_1, \ldots, v_n\} \subset \mathbf{R}^m$ $(n \leq m)$. For $1 \leq i \leq n$, let $v_i' = v_i - v_0$; and let $V' = \{v_i'\}$. If V is in general position in $\mathbf{R}^m$, then V' is linearly independent, and conversely.

15. Given $\sigma^n = v_0 v_1 \ldots v_n \subset \mathbf{R}^m$. Let

$$v = \Sigma \alpha_i v_i, \qquad w = \Sigma \beta_i v_i \in \sigma^n,$$

as in the definition of $\tau^n = \sigma^n$ in Problems 10–13. If $v = w$, then $\alpha_i = \beta_i$ for each i. (Thus it makes sense to define the barycentric coordinates of v as $(\alpha_0, \alpha_1, \ldots, \alpha_n)$.)

16. For $1 \leq j \leq m$ let E_j be the point of $\mathbf{R}^m$ with 1 as its jth coordinate, and with all other coordinates $= 0$. Thus

$$(x_1, x_2, \ldots, x_m) = \sum_{j=1}^{m} x_j E_j.$$

Given $\sigma^n = v_0 v_1 \ldots v_n$, there are numbers a_{ij} $(0 \leq i \leq n,\ 1 \leq j \leq m)$ and numbers b_j $(1 \leq j \leq m)$ such that if $v \in \sigma^n$, and

$$v = \Sigma \alpha_i v_i = \Sigma x_j E_j,$$

then

$$x_j = \Sigma_i a_{ij} \alpha_i + b_j$$

for each j. (It is in this sense that the Cartesian coordinates of v are linear functions of the barycentric coordinates of v.)

17. Let $v \in \sigma^n$, $v = \Sigma \alpha_i v_i = \Sigma x_j E_j$, as in Problem 16. Then the numbers α_i are linear functions of the numbers x_j.

18. Let K be a finite complex in $\mathbf{R}^2$, and let $\{L_i\}$ be a finite collection of lines. Then K has a subdivision K_1 in which each set $L_i \cap |K|$ forms a subcomplex.

19. Every two subdivisions K_1, K_2 of a 2-simplex $\sigma^2 \subset \mathbf{R}^2$ have a common subdivision.

20. Let K be a 2-dimensional complex (that is, a complex in which every simplex has dimension ≤ 2). Then every two subdivisions of K have a common subdivision.

21. Let K and L be complexes. If K and L are isomorphic, then there is a simplicial homeomorphism between $|K|$ and $|L|$.

22. For 2-dimensional complexes, the composition of two piecewise linear homeomorphisms is a PLH.

23. Let σ and τ be (Euclidean) simplexes, and let f be a piecewise linear homeomorphism $\sigma \to \tau$. Then $f(\sigma)$ is a simplex.

24. Let K and L be complexes. If there is a PLH between $|K|$ and $|L|$, then $K \sim_c L$; and conversely.

25. For 2-dimensional complexes, combinatorial equivalence is an equivalence relation.

26. Let K be a finite complex in $\mathbf{R}^3$, and let $\{E_i\}$ be a finite collection of planes. Then K has a subdivision in which each intersection $E_i \cap |K|$ forms a subcomplex.

27. Every two subdivisions of a 3-simplex have a common subdivision.

28. Let K be a 3-dimensional complex. Then every two subdivisions of K have a common subdivision.

29. In a topological space, if U is open, then $\operatorname{Fr} U = \bar{U} - U$.

30. Let $[X, \mathcal{O}]$ be a Hausdorff space in which every point has an open neighborhood which is homeomorphic to $\mathbf{R}$. Then $[X, \mathcal{O}]$ is separable and metrizable, and thus is a 1-manifold.

31. Let $[X, \mathcal{O}]$ and $[Y, \mathcal{O}']$ be topological spaces, and let f be a function $X \to Y$. If f is bijective and continuous, then f is a homeomorphism.

32. Every two combinatorial 2-cells are combinatorially equivalent. Similarly for combinatorial 3-cells.

33. Let $v_0 v_1 \ldots v_n$ be an n-simplex in $\mathbf{R}^n$. Then every point v of $\mathbf{R}^n$ can be represented in the form

$$v = \sum \alpha_i v_i,$$

where $\alpha_i \in \mathbf{R}$ for each i.

34. Let K be a complex. If $|K|$ is compact, then K is finite. (Of course the converse is trivial.)

Connectivity 1

A *path*, in a space $[X, \mathcal{O}]$ (or $[X, d]$) is a mapping

$$p\colon [a, b] \to X,$$

where $[a, b]$ is a closed interval in $\mathbf{R}$. If $p(a) = P$ and $p(b) = Q$, then p is a *path from P to Q*. A set $M \subset X$ is *pathwise connected* if for each two points P, Q of M there is a path $p\colon [a, b] \to M$ from P to Q (or from Q to P). If $M \subset X$, and $|p| = p([a, b]) \subset M$, then p is a *path in M*.

Theorem 1. *In a topological space $[X, \mathcal{O}]$, let G be a collection of pathwise connected sets, with a point P in common. Then the union G^* of the elements of G is pathwise connected.*

PROOF. Given $Q \in g_Q \in G$, $R \in g_R \in G$, let p be a path in g_Q, from Q to P, and let q be a path in g_R, from P to R. Then p and q fit together to give a path r, in $g_Q \cup g_R \subset G^*$, from Q to R. □

Let M and N be sets, in topological spaces $[X, \mathcal{O}]$ and $[Y, \mathcal{O}']$. A function $f\colon M \to N$ is a *mapping* if f is a mapping relative to the subspaces $[M, \mathcal{O}|M]$ and $[N, \mathcal{O}'|N]$.

Theorem 2. *Pathwise connectivity is preserved by surjective mappings. That is, if $f\colon M \twoheadrightarrow N$ is a mapping, and M is pathwise connected, then so also is N.*

PROOF. Given $P, Q \in N$, take $P', Q' \in M$ such that $f(P') = P$ and $f(Q') = Q$; and let p be a path in M from P' to Q'. Then $f(p)$ is a path in N from P to Q. □

A complex K is *connected* if it is not the union of two disjoint nonempty complexes.