Spectral Theory and Differential Operators

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PREFACE

The relationship between the classical theory of compact operators in Banach or Hilbert spaces and the study of boundary-value problems for elliptic differential equations has been a symbiotic one, each having a profound effect on the other. In the L^2 theory of elliptic differential equations with smooth coefficients and on bounded domains in $\mathbf{R}^n$, the problem of eigenfunction expansions rests upon the fact that there is a naturally occurring operator with a compact self-adjoint resolvent to which the abstract theory may be applied with great success. On the other hand, the early work of Fredholm, Hilbert, Riesz and Schmidt, for example, was stimulated by the needs of problems in integral and differential equations. The theory of compact self-adjoint operators in Hilbert space is particularly rich, but when one drops the self-adjointness substantial difficulties appear; the eigenvalues (if any) may be non-real, and, what is very important from the point of view of applications, there is no Max–Min Principle of proven usefulness for the eigenvalues; furthermore the question of whether the eigenfunctions form a basis for the underlying Hilbert space is then much more complex. When we consider compact linear operators acting in a Banach space, as is often necessary in connection with non-linear problems for example, even greater difficulties appear: to obtain information about eigenvalues indirect methods often have to be adopted. In recent years much work has been done in this area, relating eigenvalues to more geometrical quantities such as approximation numbers and entropy numbers. This work is not limited to purely abstract theory: much effort has been put into the estimation of such numbers for embedding maps between Sobolev spaces, the group in the Soviet Union led by Birman and Solomjak being especially active in this area. These embedding maps provide a natural link between the abstract theory and problems in differential (and integral) equations. Boundary-value problems for elliptic differential equations on unbounded domains or with singular coefficients necessitate the study of non-compact operators. In such cases the spectrum does not consist wholly of eigenvalues but also has a non-trivial component called the essential spectrum. In the literature there are many different ways of looking at the essential spectrum but whichever way is followed a study of Fredholm and semi-Fredholm operators is required. A notable result in this area is that due to Nussbaum and (independently) Lebow and Schechter: the radius of the essential spectrum is the same for all the commonly used definitions of essential spectrum. This brings in the notion of the measure of non-compactness of an operator, which is itself related to the entropy numbers mentioned earlier.

In order to apply the abstract theory to boundary-value problems for elliptic differential equations the first task is to determine an appropriate function space and an operator which is a natural realization of the problem. For linear elliptic problems the natural setting is an L^2 space and in this book we concentrate on the L^2 theory for general second order elliptic equations with either Dirichlet or Neumann boundary conditions.

Let

$$\tau\phi = -\sum_{i,j=1}^{n} D_i(a_{ij}D_j\phi) + \sum_{j=1}^{n} b_j D_j\phi + q\phi, \qquad D_j := \frac{\partial}{\partial x_j},$$

in an open set Ω in $\mathbf{R}^n$, with $n \geq 1$, and set

$$t[\![\phi,\psi]\!] = \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij} D_j\phi D_i\bar{\psi} + \sum_{j=1}^{n} b_j D_j\phi\bar{\psi} + q\phi\bar{\psi} \right)$$

for ϕ and ψ in $C_0^\infty(\Omega)$ or $C_0^\infty(\mathbf{R}^n)$, the choice depending on the boundary conditions under consideration. If the numerical range of t, namely the set

$$\Theta(t) = \left\{ t[\![\phi,\phi]\!] : \int_\Omega |\phi|^2 = 1 \right\},$$

lies in a sector in the complex plane with angle less than π, one can invoke the theory of sesquilinear forms to obtain an operator T whose spectrum lies within the aforementioned sector which describes the boundary-value problem associated with τ in a weak sense. If $\Theta(t)$ does not lie in a sector other techniques have to be found. In this case we can make use of the powerful methods which have been developed to tackle the problem of determining sufficiency conditions for the operator T_0 defined by a formally symmetric τ on $C_0^\infty(\Omega)$ to have a unique self-adjoint extension in $L^2(\Omega)$, a problem which has attracted a great deal of attention over the years, particularly because of its importance in quantum mechanics. An important example is Kato's distributional inequality which makes it possible to work with coefficients having minimal local requirements. Once the operator has been obtained, the next step is to analyse its spectrum. For non-self-adjoint operators the location of the various essential spectra is often as much as one can realistically hope for in the absence of the powerful tools available when the operators are self-adjoint, notably the Spectral Theorem and Max–Min Principle. Perturbation methods are effective in determining the dependence of the essential spectra on the coefficients of τ, the effect of these methods being to reduce the problem to one involving a simpler differential expression. The geometrical properties of Ω then become prominent and the properties of the embedding maps between Sobolev spaces which occur naturally achieve a special significance. In this the notion of capacity has a central role, a fact highlighted in the work of Molcanov, Maz'ja and others in the Soviet Union. To obtain information about the eigenvalues one usually has to resort to the indirect methods developed in the abstract theory. For instance, knowledge of the singular numbers of T, i.e. the eigenvalues of the non-negative self-adjoint operator $|T|$, provides information about the l^p class of the eigenvalues of T.

Our main objective in this book is to present some of the results which have been obtained during the last decade or so in connection with the problems described in the previous paragraphs. On the abstract side we deal with operators in Banach spaces whenever possible, especially as some of the most notable achievements can only be appreciated in this context. We specialize to Hilbert spaces in the work on elliptic differential equations reported on, chiefly because it is in the framework of the L^2 theory that most of the relevant recent advances have been made. Furthermore, for the L^p theory with $p \neq 2$ we have nothing substantial to add to what is contained in the books by Goldberg [1] and Schechter [2]. Despite this, when we prepare tools like the embedding theorems and results on capacity we work with L^p spaces if this can be done without much additional strain. In an area as broad as this, one is forced to be selective in one's choice of topics and, inevitably, important omissions have to be made. We say very little about eigenfunctions and expansion theorems, for instance, but we have a clear conscience about this because what we could say is adequately covered in the book by Gohberg and Krein [1]. In any case, our book is already long enough.

The book is primarily designed for the mathematician although we hope that other scientists will also find something of interest to them here and we have kept this goal in mind while writing it. The language of the book is functional analysis and a sound basic knowledge of Banach and Hilbert space theory is needed. Some familiarity with the Lebesgue integral and the elements of the theory of differential equations would be helpful but only the barest essentials are assumed. We have dispensed with a chapter of preliminaries in favour of reminders in the body of the text and where necessary we refer to other books for background material.

Most of the abstract theory is developed in the first four chapters. Chapters I and II deal with bounded linear operators in Banach spaces, the main themes being the essential spectra and the properties of various numbers like entropy numbers and approximation numbers associated with the bounded linear operators. In Chapter III closed linear operators are studied, particular emphasis being given to the behaviour of their deficiency indices and Fredholm index when the operators are extended or are perturbed. We illustrate the abstract results with a comprehensive account of general second-order quasi-differential equations and this covers the Weyl limit-point, limit-circle theory for formally symmetric equations and also its extensions by Sims and Zhikhar to formally J-self-adjoint equations. Sesquilinear forms in Hilbert spaces are the subject of Chapter IV. The basic results are the Lax-Milgram Theorem for bounded coercive forms and the representation theorems for sectorial forms. Also there are perturbation results for the forms of general self-adjoint and m-sectorial operators which have an important role to play later in the location of the essential spectra of differential operators. Another result which will be important later is Stampacchia's generalization of the Lax–Milgram theorem to variational inequalities.

In Chapter V we give a treatment of Sobolev spaces. Apart from their intrinsic interest these spaces are an indispensible tool for any work on partial differential equations and much of what is done in subsequent chapters hinges on Chapter V. Furthermore Sobolev spaces are an ideal testing ground for examining some of the abstract notions discussed in the early chapters and accordingly we devote some space to the determination of the measures of non-compactness and the approximation numbers of embedding maps between Sobolev spaces.

The remaining chapters deal mainly with second-order elliptic differential operators. The weak or generalized forms of the Dirichlet and Neumann boundary-value problems are defined and studied in Chapter VI. The material in Chapter VI is mainly relevant to bounded open sets Ω in $\mathbb{R}^n$ when the underlying operators have compact resolvents in $L^2(\Omega)$, in which case the spectra consist wholly of eigenvalues. Also included is Stampacchia's weak maximum principle and this leads naturally to the notion of capacity. Second-order operators on arbitrary open sets Ω are the theme of Chapter VII. Under weak conditions on the coefficients of the differential expression we describe three different techniques for determining the Dirichlet and Neumann operators. The first applies the First Representation Theorem to sectorial forms, the second is one developed by Kato based on his celebrated distributional inequality and the third has its roots in the work of Levinson and Titchmarsh on the essential self-adjointness of the operator defined by $-\Delta+q$ on $C_0^\infty(\Omega)$ when q is real. Schrödinger operators are an important special case, especially of the third class of operators discussed, and some of the results obtained for highly oscillatory potentials are anticipated by the quantum-mechanical interpretation of the problem.

The central result of Chapter VIII is Molcanov's necessary and sufficient condition for the self-adjoint realization of $-\Delta+q$ (q real and bounded below) to have a wholly discrete spectrum. This necessitates the study of capacity and in the wake of the main result we also obtain necessary and sufficient conditions for the embedding $W_0^{1,p}(\Omega) \to L^p(\Omega)$ to be compact and for important integral inequalities (like the Poincaré inequality) to hold.

In Chapter IX we study the essential spectra of closed operators in Banach and Hilbert spaces and then use the abstract theory to locate the various essential spectra of constant coefficient differential operators in $L^2(\mathbb{R}^n)$ and $L^2(0,\infty)$. In the case when the coefficients are not constant a useful tool for ordinary differential operators is the so-called Decomposition Principle which implies that the essential spectra depend only on the behaviour of the coefficients at infinity. For partial differential operators a Decomposition Principle is obtained in Chapter X as a perturbation result and this is then used to locate the essential spectra of the general second-order operators in $L^2(\Omega)$ discussed in Chapter VII. We analyse the dependence of the essential spectra on Ω in two different ways. In the first the results are described in terms of capacity and sequences of cubes which intersect Ω. The second involves the use of a mean distance function $m(x)$, which is a measure of the distance of x to the boundary of Ω, and an integral inequality obtained by E. B. Davies. This enables us to give estimates for the first eigenvalue and the least point of the essential spectrum of the Dirichlet problem for $-\Delta$ on Ω.

The last two chapters are concerned with the eigenvalues and singular values of the Dirichlet and Neumann problems for $-\Delta + q$. The case of q real, and hence self-adjoint operators, is treated in Chapter XI, the main result being a global estimate for $N(\lambda)$, the number of eigenvalues less than λ when λ is below the essential spectrum. From this estimate asymptotic formulae are derived for $N(\lambda)$ when the spectrum is discrete and $\lambda \to \infty$ and when the negative spectrum is discrete and $\lambda \to 0-$. We also obtain the Cwikel–Lieb–Rosenblyjum estimate for $N(\lambda)$ when $q \in L^{n/2}(\mathbb{R}^n)$ with $n \geqslant 3$, and include the elegant Li–Yau proof of the latter result. In Chapter XII q is complex, and global and asymptotic estimates are obtained for $M(\lambda)$, the number of singular values less than λ. From these estimates the l^p-class of the singular numbers and eigenvalues are derived.

Chapters are divided into sections, and some sections into subsections. For example, §I.3.2 means subsection 2 of section 3 of Chapter I; it is simply written §3.2 when referred to within the same chapter and §2 when referred to within the same section. Theorems, Corollaries, Lemmas, Propositions, and Remarks are numbered consecutively within each section. Theorem I.2.3 means Theorem 2.3 in §2 of Chapter I and is referred to simply as Theorem 2.3 within the same chapter. Formulae are numbered consecutively within each section; (I.2.3) means the third equation of §2 of Chapter I and is referred to as (2.3) within the same chapter. The symbol ■ indicates the end of the statement of a result and □ indicates the end of a proof.

There is also a glossary of terms and notation, a bibliography and an index.

We have made no systematic attempt to go into the complicated history of the results presented here, but hope that the references provided will be helpful to the reader interested in the background of the material.

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Basic notation

$B(x, r)$: open ball in $\mathbf{R}^n$, centre x and radius r.
$\mathbf{C}$: complex plane; $\mathbf{C}_{\pm} = \{z \in \mathbf{C}: \operatorname{im} z \gtrless 0\}$; $\mathbf{C}^n$: n-dimensional complex space; $\mathbf{R}$: real line; $\mathbf{R}^n$: n-dimensional Euclidean space.
$\mathbf{R}^n_+ = \mathbf{R}^n \setminus \{0\}$.
$D_i u = \partial u / \partial x_i$; if $\alpha = (\alpha_1, \ldots, \alpha_n)$ with α_i non-negative integers, $D^\alpha u = \partial^{|\alpha|} u / \partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}$, where $|\alpha| = \alpha_1 + \ldots + \alpha_n$.
Ω: an open set in $\mathbf{R}^n$; Ω is a *domain* if it is also connected.
$\partial\Omega$: boundary of Ω; $\bar{\Omega}$: closure of Ω; $\Omega^c = \mathbf{R}^n \setminus \Omega$.
$\Omega' \subset\subset \Omega$: $\bar{\Omega}'$ is a compact subset of Ω.
dist $(x, \partial\Omega)$: distance from x to Ω^c.
$\mathbf{N}$: positive integers; $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$; $\mathbf{Z}$: all integers.
$f(t) \asymp g(t)$ as $t \to a$: there exist positive constants c_1, c_2 such that $c_1 \leqslant f(t)/g(t) \leqslant c_2$ for $|t - a|$ $(\neq 0)$ small enough, if $a \in \mathbf{R}$; and for large enough $\pm t$ if $a = \pm\infty$.
$T \restriction_G$: restriction of the operator (or function) T to set G.
$f^+ = \max(f, 0)$, $f^- = -\min(f, 0)$.
$A \subset B$ for sets A, B allows for $A = B$.
Embedding: a bounded linear injective map of a Banach space X to another such space Y.
$l^p (1 \leqslant p \leqslant \infty)$: complex sequence space with norm $\|(\xi_j)\|_p = (\Sigma |\xi_j|^p)^{1/p}$ when $1 \leqslant p < \infty$ and $\|(\xi_j)\|_\infty = \sup_j |\xi_j|$ when $p = \infty$.

$c_0 := \{(\xi_j) \in l^\infty : \lim_j \xi_j = 0\}$.

ω_n: volume of the unit ball in $\mathbf{R}^n$, i.e. $\omega_n = \dfrac{\pi^{n/2}}{\Gamma(1 + \frac{1}{2}n)}$.

CONTENTS

I

Linear operators in Banach spaces

Three main themes run through this chapter: compact linear operators, measures of non-compactness, and Fredholm and semi-Fredholm maps. Each topic is of considerable intrinsic interest; our object is not only to make this apparent but also to establish connections between the themes so as to derive results which will be of great interest later. One such result is a formula for the radius $\mathrm{r_e}(T)$ of the essential spectrum of a bounded linear map T.

The theory of compact linear operators acting in a Banach space has a classical core which will be familiar to many, and in view of this we pass rather quickly over it. Perhaps less well-known is the concept of the measure of non-compactness of a set and of a map, a notion due to Kuratowski [1], who introduced it in 1930 for subsets of a metric space. The idea lay more or less dormant until 1955, when Darbo[1] showed how it could be used to obtain a significant generalization of Schauder's fixed-point theorem. Since that time, substantial advances have been made both in the theory and in applications, although the bulk of applications have been to ordinary rather than to partial differential equations. We try to redress the balance later on in the book by use of the formula for $\mathrm{r_e}(T)$ in our discussion of the essential spectrum of various partial differential operators. The interaction between measures of non-compactness and semi-Fredholm maps is of crucial importance in the derivation of this formula, and accordingly we devote some time to this interplay.

1. Compact linear maps

All vector spaces which will be mentioned will be assumed to be over the complex field, unless otherwise stated. The norm on a normed vector space X will usually be denoted by $\|\bullet\|_X$, or by $\|\bullet\|$ if no ambiguity is possible.

Given any Banach spaces X and Y, the vector space of all bounded linear maps from X to Y will be denoted by $\mathscr{B}(X,Y)$, or by $\mathscr{B}(X)$ if $X = Y$; with the norm $\|\bullet\|$ defined by $\|T\| = \sup\{\|Tx\|:\|x\| \leqslant 1\}$, $\mathscr{B}(X,Y)$ is a Banach space. It is natural to try to distinguish members of $\mathscr{B}(X,Y)$ which have particularly good properties. Compact linear maps come into this category, since they have properties reminiscent of linear maps acting between finite-dimensional spaces.

Definition 1.1. Let X and Y be Banach spaces and let $T: X \to Y$ be linear. The map T is said to be *compact* if, and only if, for any bounded subset B of X, the closure $\overline{T(B)}$ of $T(B)$ is compact. ■

Evidently T is compact if, and only if, given any bounded sequence (x_n) in X, the sequence (Tx_n) contains a convergent subsequence. Note also that if T is compact it is continuous, since otherwise there would be a sequence (x_n) in X such that $\| x_n \| = 1$ for all $n \in \mathbb{N}$, and $\| Tx_n \| \to \infty$ as $n \to \infty$, which is impossible.

Examples. (i) If $T \in \mathscr{B}(X,Y)$ and the dimension of the range $\mathscr{R}(T) := T(X)$ of T, $\dim \mathscr{R}(T)$, is finite, then T must be compact, since if B is a bounded subset of X then $\overline{T(B)}$ is closed and bounded, and hence compact.

(ii) Not every bounded linear map is compact: take $X = Y = l^2$, for each $n \in \mathbb{N}$ let $e^{(n)}$ be the element of l^2 with mth coordinate δ_{mn} (equal to 1 if $m = n$, and 0 otherwise), and observe that the identity map of l^2 to itself is continuous but not compact, because the sequence $(e^{(n)})$ has no convergent subsequence.

(iii) Let $a, b \in \mathbb{R}$, $b > a$, $J = [a, b]$, and suppose that $k: J \times J \to \mathbb{C}$ is continuous on $J \times J$; define

$$(Kx)(s) = \int_a^b k(s, t)\, x(t)\, \mathrm{d}t$$

for all $s \in J$ and for all x in the Banach space $C(J)$ of all continuous complex-valued functions on J (with norm $\| \bullet \|$ given by $\| x \| = \max \{ |x(s)| : s \in J\}$). Then K is a linear map of $C(J)$ into itself, and in fact, K is compact. To see this, set $M = \max \{ |k(s, t)| : s, t \in J\}$; then $\| Kx \| \leqslant M(b-a) \| x \|$ for all $x \in C(J)$, and so if B is a bounded subset of $C(J)$ then $K(B)$ is bounded. Moreover, for all $s_1, s_2 \in J$ and any $x \in C(J)$,

$$\begin{aligned} |(Kx)(s_1) - (Kx)(s_2)|^2 &= \left| \int_a^b [k(s_1, t) - k(s_2, t)]\, x(t)\, \mathrm{d}t \right|^2 \\ &\leqslant \int_a^b |k(s_1, t) - k(s_2, t)|^2\, \mathrm{d}t \cdot \int_a^b |x(t)|^2\, \mathrm{d}t . \end{aligned}$$

Thus given any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|(Kx)(s_1) - (Kx)(s_2)| < \varepsilon$ if $\| x \| \leqslant 1$ and $|s_1 - s_2| < \delta$ $(s_1, s_2 \in J)$. Hence $\{Kx : x \in C(J),\ \| x \| \leqslant 1\}$ is equicontinuous and bounded, and thus relatively compact, by the Arzela–Ascoli Theorem (cf. Yosida [1, p. 85]).

Denote by $\mathscr{K}(X,Y)$ the family of all compact linear maps from X to Y, and put $\mathscr{K}(X) = \mathscr{K}(X, X)$. The following proposition is a well-known consequence of the definition of compactness.

Proposition 1.2. Let X,Y,Z be Banach spaces. Then $\mathscr{K}(X,Y)$ is a closed linear subspace of $\mathscr{B}(X,Y)$; if $T_1 \in \mathscr{B}(X,Y)$ and $T_2 \in \mathscr{B}(Y,Z)$ then $T_2 T_1$ is compact if either T_1 or T_2 is compact. ■

This proposition implies that $\mathscr{K}(X)$ is a closed two-sided ideal in the Banach algebra $\mathscr{B}(X)$.

It has already been noted that if $T \in \mathscr{B}(X, Y)$ is of *finite rank*, that is, $\dim \mathscr{R}(T) < \infty$, then $T \in \mathscr{K}(X,Y)$. In particular, $\mathscr{B}(X,Y) = \mathscr{K}(X,Y)$ if either X or Y is finite-dimensional. The following result complements this, and throws new light on Example (ii) above.

Theorem 1.3. Let X be a Banach space and suppose that the identity map of X to itself is compact. Then $\dim X < \infty$. ■

This follows directly from the following lemma.

Lemma 1.4. Let (X_n) be a sequence of finite-dimensional linear subspaces of a Banach space X such that for all $n \in \mathbb{N}$, $X_n \subset X_{n+1}$ and $X_n \neq X_{n+1}$. Then given any $n \in \mathbb{N}$ with $n \geqslant 2$, there exists $x_n \in X_n$, with $\|x_n\| = 1$, such that $\|x_n - x\| \geqslant 1$ for all $x \in X_{n-1}$. In particular, $\| x_n - x_m \| \geqslant 1$ when $m < n$; the sequence (x_n) has no convergent subsequences. ■

Proof. Let $y_n \in X_n \setminus X_{n-1}$. The function $y \mapsto \| y - y_n \|$ is positive and continuous on X_{n-1} and approaches infinity as $\|y\| \to \infty$; hence it has a minimum, at $z_n \in X_{n-1}$, say, and

$$0 < \| z_n - y_n \| \leqslant \| x + z_n - y_n \|$$

for any $x \in X_{n-1}$. The point $x_n := (y_n - z_n)/\| y_n - z_n \|$ then has all the required properties. □

A related result is the following.

Lemma 1.5 (Riesz's Lemma). Let M be a proper, closed, linear subspace of a normed vector space X. Then given any $\theta \in (0, 1)$, there is an element $x_\theta \in X$ such that $\| x_\theta \| = 1$ and $\text{dist}\,(x_\theta, M) \geqslant \theta$. ■

Proof. Let $x \in X \setminus M$. Since M is closed, $d := \text{dist}\,(x, M) > 0$. Thus given any $\theta \in (0, 1)$, there exists $m_\theta \in M$ such that $\| x - m_\theta \| \leqslant d/\theta$. The element $x_\theta :=$ $(x - m_\theta)/\| x - m_\theta \|$ has all the properties needed. □

Compactness of a linear map is preserved by the taking of the adjoint. Before this result is given in a formal way, some remarks about notation are desirable. Given any normed vector space X, by the *adjoint space* X^* of X is meant the set of all *conjugate* linear continuous functionals on X; that is, $f \in X^*$ if, and only if, $f : X \to \mathbb{C}$ is continuous and $f(\alpha x + \beta y) = \bar{\alpha} f(x) + \bar{\beta} f(y)$ for all $\alpha,\beta \in \mathbb{C}$ and all $x,y \in X$. Our choice of *conjugate-linear* functionals, rather than the more common *linear* functionals, is dictated solely by the convenience which will result later on in the book. With the usual definitions of addition and multiplication by scalars the adjoint space X^* becomes a Banach space when given the norm $\|\bullet\|$ defined by

$$\|f\| = \sup\{|(f,x)| : \|x\| = 1\},$$

where (f,x) is the value of f at x (often denoted by $f(x)$ as above. Strictly speaking, we should write $(f,x)_X$, but the subscript will be omitted if no ambiguity is possible. The same omission will be made for inner products in a Hilbert space.). Given any $T \in \mathscr{B}(X,Y)$, the adjoint of T is the map $T^* \in \mathscr{B}(Y^*, X^*)$ defined by $(T^*g, x) = (g, Tx)$ for all $g \in Y^*$ and $x \in X$; note that $(\alpha S + \beta T)^* = \bar{\alpha}S^* + \bar{\beta}T^*$ for all $\alpha, \beta \in \mathbb{C}$ and all $S, T \in \mathscr{B}(X,Y)$. These conventions about adjoints will apply even when the underlying spaces are Hilbert spaces, and there will therefore be none of the usual awkwardness about the distinction between Banach-space and Hilbert-space adjoints of a map that has to be made when linear, rather than conjugate-linear, functionals are used. Of course, many of the results to be given below would also hold had X^* been defined to be the space of all continuous linear functionals on X. Note that the Riesz Representation Theorem (cf. Taylor[1, Theorem 4.81-C]) enables any Hilbert space H to be identified with H^*; and that in view of this, given any $T \in \mathscr{B}(H_1, H_2)$ (H_1 and H_2 being Hilbert spaces), $T^* \in \mathscr{B}(H_2, H_1)$. If $H_1 = H_2 = H$, then both T and T^* belong to $\mathscr{B}(H)$: the map T is said to be *self-adjoint* if $T = T^*$.

Theorem 1.6. Let X and Y be Banach spaces and let $T \in \mathscr{B}(X,Y)$. Then $T \in \mathscr{K}(X,Y)$ if, and only if, $T^* \in \mathscr{K}(Y^*, X^*)$. ■

The well-known proof may be found in Yosida [1, p. 282].

The notion of the adjoint of a map T will also be needed when T is unbounded. Thus let $\mathscr{D}(T)$ be a linear subspace of X which is *dense* in X (i.e. $\overline{\mathscr{D}(T)} = X$) and let $T: \mathscr{D}(T) \to Y$ be linear. Let $\mathscr{D}(T^*) = \{g \in Y^*$: there exists $f \in X^*$ such that $(f,x) = (g,Tx)$ for all $x \in \mathscr{D}(T)\}$; the adjoint T^* of T is the map $T^*: \mathscr{D}(T^*) \to X^*$ defined by $T^*g = f$, i.e., $(T^*g, x) = (g, Tx)$ for all $x \in \mathscr{D}(T)$ and all $g \in \mathscr{D}(T^*)$. Note that it is essential that $\mathscr{D}(T)$ be dense in X for T^* to be well-defined. A more detailed discussion of self-adjoint maps in a Hilbert space will be given in §III.4.

Next, the results of the Fredholm–Riesz–Schauder theory of compact linear maps will be given; this theory extends in a most direct way the theory of linear maps in finite-dimensional spaces. The complete picture follows from a series of auxiliary results, a number of which are of interest in their own right. Throughout the discussion X will stand for a non-trivial (that is $\neq \{0\}$) Banach space, I will be the identity map from X to X; and given any $T \in \mathscr{B}(X)$ and any $\lambda \in \mathbb{C}$, we shall write T_λ for $T - \lambda I$. The notions of the *resolvent set* and the *spectrum* of a linear map will also be needed; these will be explained in terms of a linear map S from a linear subspace $\mathscr{D}(S)$ of X to X. The resolvent