PERTURBATION THEORY OF EIGENVALUE PROBLEMS

Franz Rellich

notes on mathematics and its applications

GORDON AND BREACH

Perturbation Theory of Eigenvalue Problems

Franz RELLICH
late of
Institute of Mathematical Sciences
New York University

assisted by
J. Berkowitz

GORDON AND BREACH
Science Publishers

NEW YORK LONDON PARIS

Library of Congress catalog card number: 68-21967

Editorial office for the United Kingdom:
Gordon and Breach Science Publishers Ltd.
8 Bloomsbury Way
London W.C.1

Editorial office for France:
Gordon & Breach
7-9 rue Emile Dubois
Paris 14e

Distributed in Canada by:
The Ryerson Press
299 Queen Street West
Toronto 2B, Ontario

General Preface

A LARGE number of mathematics books begin as lecture notes; but, since mathematicians are busy, and since the labor required to bring lecture notes up to the level of perfection which authors and the public demand of formally published books is very considerable, it follows that an even larger number of lecture notes make the transition to book form only after great delay or not at all. The present lecture note series aims to fill the resulting gap. It will consist of reprinted lecture notes, edited at least to a satisfactory level of completeness and intelligibility, though not necessarily to the perfection which is expected of a book. In addition to lecture notes, the series will include volumes of collected reprints of journal articles as current developments indicate, and mixed volumes including both notes and reprints.

JACOB T. SCHWARTZ
MAURICE LÉVY

Preface

THE PRESENT notes, taken from lectures given by Professor Rellich at New York University in 1953, describe an area which he pioneered, and in which some of his most striking mathematical work was done. They show the fruitful interplay, characteristic for Rellich, of abstract operator theory which penetrating investigations of significant particular examples. In reading these notes, one comes to exceptionally close witness of the mathematical mind in the act of creation. It is to be hoped that their reissue serves, in tribute to the author, to advance the theory with which they are concerned.

JACOB T. SCHWARTZ

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Introduction

Perturbation methods attempt to solve a given problem by approximating it by simpler problems whose solutions are more or less explicitly known. In eigenvalue problems the perturbation method yields numerical results comparatively quickly provided you are satisfied with approximations of low order. However, even in problems which appear to be very simple, it might be difficult to ascertain whether or not the method applied would converge if continued ad infinitum or to estimate the error incurred by stopping at a certain order of approximation. Sometimes the method obviously does not converge—at least not in the usual sense; then there is the problem of trying to interpret the results computed, if they have any significance at all. The following examples illustrate the mathematical problems involved in these questions.

§ 1. A Small Perturbation Parameter does not mean a Small Perturbation

We consider the eigenvalue problem

$$\int_{-\infty}^{\infty} e^{-|x|-|y|}\, u(y)\, dy = \lambda u(x), \quad -\infty < x < +\infty .$$

By a *point eigenvalue* we mean any number λ for which there exists a (possibly complex-valued) solution $u(x) \not\equiv 0$ of this integral equation such that $\int_{-\infty}^{\infty} |u|^2\, dx < \infty$. (Integration here is understood to be in the sense of Lebesgue.) The function $u(x)$ is called an *eigenfunction* belonging to λ. We use the name "point eigenvalue" rather than just "eigenvalue" to emphasize the fact that we do not mean the points of the continuous spectrum (which happens to be absent in this simple example). It is a simple matter to find all the point eigenvalues in this example. Suppose λ is an eigenvalue; then

$$\lambda u(x) = c\, e^{-|x|}$$

where

$$c = \int_{-\infty}^{\infty} e^{-|y|}\, u(y)\, dy.$$

Hence

$$\lambda \int_{-\infty}^{\infty} e^{-|x|}\, u(x)\, dx = c \int_{-\infty}^{\infty} e^{-2|x|}\, dx = c,$$

and consequently

$$\lambda c = c.$$

If $c \neq 0$ then $\lambda = 1$ and $\phi_0(x) = e^{-|x|}$ is an eigenfunction belonging to $\lambda = 1$. If $c = 0$ then $\lambda = 0$, for otherwise $u(x) \equiv 0$ which is not possible for an eigenfunction. Thus every function $\phi(x)$ with $\phi(x) \not\equiv 0$, $\int_{-\infty}^{\infty} |\phi|^2\, dx < \infty$, and $\int_{-\infty}^{\infty} e^{-|x|}\, \phi(x)\, dx = 0$ is an eigenfunction belonging to the point eigenvalue $\lambda = 0$. An arbitrary function $f(x)$ with $\int_{-\infty}^{\infty} |f|^2\, dx < \infty$ can be "expanded" in the form

$$f(x) = f_0 \phi_0(x) + \phi(x),$$

with

$$f_0 = \int_{-\infty}^{\infty} \phi_0(x) f(x)\, dx,$$

where ϕ_0 and ϕ are eigenfunctions belonging to $\lambda = 1$ and to $\lambda = 0$, respectively. The point eigenvalue $\lambda = 1$ is simple (non-degenerate) whereas $\lambda = 0$ is of infinite multiplicity.

We now turn to the eigenvalue problem

$$\int_{-\infty}^{\infty} e^{-|x|-|y|}\, u(y)\, dy + \varepsilon x u(x) = \lambda u(x), \quad -\infty < x < +\infty,$$

with ε a given real number. Assuming ε small, we seek a point eigenvalue $\lambda = \lambda(\varepsilon)$ and a corresponding eigenfunction $u = u(x; \varepsilon)$ of the form

$$\lambda = \lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2 + \cdots, \qquad \lambda_0 = 1;$$

$$u = \phi_0 + \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \cdots, \quad \phi_0 = e^{-|x|}.$$

If we introduce the abbreviation

$$Au = \int_{-\infty}^{\infty} e^{-|x|-|y|}\, u(y)\, dy, \quad Bu = x u(x),$$

we have

$$(A + \varepsilon B)(\phi_0 + \varepsilon \phi_1 + \cdots) = (\lambda_0 + \varepsilon \lambda_1 + \cdots)(\phi_0 + \varepsilon \phi_1 + \cdots).$$

If the convergence of $\phi_0 + \varepsilon\phi_1 + \cdots$ is good enough to permit the term by term multiplication

$$(A + \varepsilon B)(\phi_0 + \varepsilon\phi_1 + \cdots) = A\phi_0 + \varepsilon\,(B\phi_0 + A\phi_1) + \cdots,$$

then we obtain the equations:

$$\begin{aligned}
A\phi_0 - \lambda_0\phi_0 &= 0,\\
A\phi_1 - \lambda_0\phi_1 &= \lambda_1\phi_0 - B\phi_0,\\
A\phi_2 - \lambda_0\phi_2 &= \lambda_2\phi_0 + \lambda_1\phi_1 - B\phi_1,\\
&\cdots\cdots\cdots\cdots\\
A\phi_n - \lambda_0\phi_n &= \lambda_n\phi_0 + \lambda_{n-1}\phi_1 + \cdots + \lambda_1\phi_{n-1} - B\phi_{n-1}\\
&\cdots\cdots\cdots\cdots
\end{aligned}$$

The first equation provides no new information; it is satisfied by $\lambda_0 = 1$, $\phi_0 = e^{-|x|}$. The remaining equations are of the form

$$A\phi_n - \lambda_0\phi_n = f, \qquad n = 1, 2, \ldots;$$

consequently we have

$$(\phi_0, A\phi_n - \lambda_0\phi_n) = (\phi_0, f),$$

using here the "inner product"

$$(g, f) = \int_{-\infty}^{\infty} \overline{g(x)}\, f(x)\, dx.$$

Evidently

$$(\phi_0, A\phi_n) = (A\phi_0, \phi_n)$$

because the kernel $e^{-|x|-|y|}$ of the integral operator A is symmetric in x and y. Thus

$$(\phi_0, f) = (\phi_0, A\phi_n) - \lambda_0\,(\phi_0, \phi_n) = 0,$$

and therefore

$$\lambda_n(\phi_0, \phi_0) + \lambda_{n-1}(\phi_0, \phi_1) + \cdots + \lambda_1(\phi_0, \phi_{n-1}) - (\phi_0, B\phi_{n-1}) = 0, \qquad n = 1, 2, \ldots.$$

For $n = 1$, we find

$$\lambda_1 = (\phi_0, B\phi_0),$$

or in words:

The first order approximation to the eigenvalue is the mean value of the perturbation operator B with respect to the unperturbed normalized eigenfunction ϕ_0.

In our special case we have

$$\lambda_1 = \int_{-\infty}^{\infty} x\, e^{-2|x|}\, dx = 0.$$

The first order approximation $\phi_1(x)$ can be determined from the equation

$$A\phi_1 - \lambda_0\phi_1 = \lambda_1\phi_0 - B\phi_0;$$

the right hand side is now known. We find

$$c_1\, e^{-|x|} - \phi_1 = -x\, e^{-|x|}$$

where

$$c_1 = \int_{-\infty}^{\infty} e^{-|x|}\, \phi_1(x)\, dx,$$

and thus

$$\phi_1 = (x + c_1)\, e^{-|x|}.$$

Evidently this function is a solution of $A\phi_1 - \lambda_0\phi_1 = \lambda_1\phi_0 - B\phi_0$ no matter what the value of c_1. If we require that the function $u = \phi_0 + \varepsilon\phi_1 + \cdots$ be normalized, we must have

$$1 = (\phi_0 + \varepsilon\phi_1 + \cdots, \phi_0 + \varepsilon\phi_1 + \cdots),$$

which gives rise to the following infinitely many equations:

$$1 = (\phi_0, \phi_0),$$

$$0 = (\phi_0, \phi_1) + (\phi_1, \phi_0),$$

$$\cdots\cdots\cdots\cdots$$

$$0 = (\phi_0, \phi_n) + (\phi_1, \phi_{n-1}) + \cdots + (\phi_n, \phi_0),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

In addition we require all the functions $\phi_1, \phi_2, \ldots$ to be real. Then

$$0 = (\phi_0, \phi_1) + (\phi_1, \phi_0) = 2\,(\phi_0, \phi_1),$$

and hence

$$0 = (\phi_0, \phi_1) = \int_{-\infty}^{\infty} (x + c_1)\, e^{-2|x|}\, dx = c_1.$$

Thus

$$\phi_1(x) = x\, e^{-|x|}.$$

Since

$$\lambda_2\,(\phi_0, \phi_0) + \lambda_1\,(\phi_0, \phi_1) - (\phi_0, B\phi_1) = 0,$$

we obtain

$$\lambda_2 = \tfrac{1}{2}.$$

Next we use the equation

$$A\phi_2 - \lambda_0\phi_2 = \lambda_2\phi_0 + \lambda_1\phi_1 - B\phi_1$$

which means

$$e^{-|x|}c_2 - \phi_2 = \tfrac{1}{2}e^{-|x|} - x^2e^{-|x|}$$

where

$$c_2 = \int_{-\infty}^{\infty} e^{-|x|}\phi_2(x)\,dx.$$

Thus

$$\phi_2(x) = (x^2 + c_2 - \tfrac{1}{2})\,e^{-|x|}.$$

From the equation

$$0 = (\phi_0, \phi_2) + (\phi_1, \phi_1) + (\phi_2, \phi_0)$$

and the fact that ϕ_0, ϕ_1, and ϕ_2 are all real, we find

$$(\phi_0, \phi_2) = -\tfrac{1}{2}\int_{-\infty}^{\infty} x^2e^{-2|x|}\,dx = -\tfrac{1}{4}.$$

Hence

$$c_2 - \tfrac{1}{2} = -\tfrac{3}{4},$$

and

$$\phi_2(x) = (x^2 - \tfrac{3}{4})\,e^{-|x|}.$$

Thus we obtain

$$\lambda = 1 + \tfrac{1}{2}\varepsilon^2 + \cdots,$$

$$u = e^{-|x|}\,(1 + x\varepsilon + (x^2 - \tfrac{3}{4})\,\varepsilon^2 + \cdots).$$

Questions: Do these series converge, if we compute successively the coefficients? Do these series provide a solution of the equation

$$(A + \varepsilon B)\,u = \lambda u$$

with

$$\int_{-\infty}^{\infty} |u|^2\,dx = 1\,?$$

Unfortunately, the answers must be in the negative because our equation has no (real) point eigenvalue at all if ε is a real number not zero. Indeed if a solution $u(x) \not\equiv 0$ with $\int_{-\infty}^{\infty} |u|^2\,dx < \infty$ were to exist, then

$$-c = \int_{-\infty}^{\infty} e^{-|y|}\,u(y)\,dy$$

would exist and we should have

$$-c\,e^{-|x|} + \varepsilon x u\,(x) = \lambda u\,(x).$$

Since ε and λ are real, $\varepsilon \neq 0$, it would then follow that

$$u(x) = \frac{c\, e^{-|x|}}{\varepsilon x - \lambda}$$

for $x \neq \dfrac{\lambda}{\varepsilon}$. The singularity at $x = \dfrac{\lambda}{\varepsilon}$ is a pole of order 1. Then $\int_{-\infty}^{\infty} |u|^2\, dx < \infty$ is only possible if $c = 0$. But $c = 0$ would imply $u(x) \equiv 0$ which is not possible if u were an eigenfunction. As a matter of fact the spectrum of the operator $A + \varepsilon B$ is purely continuous.

The reason for the disappearance of the point eigenvalue $\lambda = 1$, present in $A + \varepsilon B$ for $\varepsilon = 0$ but vanished for $\varepsilon \neq 0$ no matter how small $|\varepsilon|$, lies in the fact that the coefficient εx in the perturbation term $\varepsilon Bu = \varepsilon xu$ is never small in the whole interval $-\infty < x < \infty$ unless $\varepsilon = 0$.

Suppose, instead of εxu, we consider the perturbation term $\varepsilon s(x)\, u$ where $s(x)$ is a *bounded* piecewise† continuous function in $-\infty < x < \infty$. Then we deal with the eigenvalue problem

$$\int_{-\infty}^{\infty} e^{-|x|-|y|} u(y)\, dy + \varepsilon s(x)\, u(x) = \lambda u(x), \qquad -\infty < x < \infty.$$

Again we set

$$\lambda = \lambda_0 + \varepsilon \lambda_1 + \cdots, \qquad \lambda_0 = 1,$$

$$u = \phi_0 + \varepsilon \phi_1 + \cdots, \quad \phi_0 = e^{-|x|};$$

and we successively compute $\lambda_1, \phi_1, \ldots$ as before. For example, we find

$$\lambda_1 = \int_{-\infty}^{\infty} s(x)\, e^{-2|x|} dx, \quad \phi_1 = (s(x) - \lambda_1)\, e^{-|x|}.$$

Now we can prove that a point eigenvalue $\lambda = \lambda(\varepsilon)$ exists which can be represented as a convergent power series

$$\lambda = 1 + \varepsilon \lambda_1 + \cdots$$

for small $|\varepsilon|$.

† A function $f(x)$ is said to be piecewise continuous in a closed bounded interval $a \leq x \leq b$ when $f(x)$ is continuous there with the possible exception of a finite number of points x_i at which $\lim\limits_{\substack{x \to x_i \\ x < x_i}} f(x)$ and $\lim\limits_{\substack{x \to x_i \\ x > x_i}} f(x)$ exist and are finite. A function $f(x)$ is said to be piecewise continuous in an open interval $\alpha < x < \beta$ if it is piecewise continuous in every closed bounded subinterval of $\alpha < x < \beta$.

Assuming temporarily the existence of a point eigenvalue λ, we have for a corresponding eigenfunction u the equation

$$u(x) = \frac{c\, e^{-|x|}}{\lambda - \varepsilon s(x)}$$

where

$$c = \int_{-\infty}^{\infty} e^{-|x|} u(x)\, dx .$$

If $u(x) \not\equiv 0$ then $c \neq 0$; and we obtain the equation

$$1 = \int_{-\infty}^{\infty} \frac{e^{-2|x|}}{\lambda - \varepsilon s(x)}\, dx .$$

Forgetting the heuristic derivation of this equation, we shall solve it for $\lambda = \lambda(\varepsilon)$ and find $\lambda = 1 + \lambda_1 \varepsilon + \cdots$ as a power series convergent for small $|\varepsilon|$. Suppose this were done. Then for small $|\varepsilon|$ the expression $\lambda(\varepsilon) - \varepsilon s(x)$ does not vanish in $-\infty < x < \infty$, and the function

$$u = u(x; \varepsilon) = \frac{c\, e^{-|x|}}{\lambda(\varepsilon) - \varepsilon s(x)}$$

for any constant c is such that

$$\int_{-\infty}^{\infty} |u|^2 dx < \infty .$$

Furthermore, for this function u we have

$$\begin{aligned} &\int_{-\infty}^{\infty} e^{-|x|-|y|} u(y)\, dy + \varepsilon s(x)\, u(x) \\ &\quad = e^{-|x|} \int_{-\infty}^{\infty} \frac{c\, e^{-2|y|}}{\lambda(\varepsilon) - \varepsilon s(y)}\, dy + \frac{\varepsilon c s(x)\, e^{-|x|}}{\lambda(\varepsilon) - \varepsilon s(x)} \\ &\quad = c\, e^{-|x|} + \frac{\varepsilon c s(x)\, e^{-|x|}}{\lambda(\varepsilon) - \varepsilon s(x)} = \lambda(\varepsilon) \frac{c\, e^{-|x|}}{\lambda(\varepsilon) - \varepsilon s(x)} = \lambda(\varepsilon)\, u . \end{aligned}$$

In other words, u satisfies the eigenvalue equation with $\lambda = \lambda(\varepsilon)$. Therefore $\lambda(\varepsilon)$ is indeed a point eigenvalue.

The only gap that must be filled is the existence of a convergent power series solution

$$\lambda = \lambda(\varepsilon) = 1 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots$$

of the functional equation

$$1 = \int_{-\infty}^{\infty} \frac{e^{-2|x|}}{\lambda - \varepsilon s(x)} \, dx.$$

We make the change of variable $\mu = \dfrac{\varepsilon}{\lambda}$ to obtain the equation

$$\varepsilon = \mu \int_{-\infty}^{\infty} \frac{e^{-2|x|}}{1 - \mu s(x)} \, dx.$$

If $|\mu| < \dfrac{1}{2s_0}$ where s_0 is an upper bound of $|s(x)|$ for all x, then the integrand of the above integral can be expanded into a convergent series and termwise integration yields

$$\int_{-\infty}^{\infty} \frac{e^{-2|x|}}{1 - \mu s(x)} \, dx = 1 + \sum_{n=1}^{\infty} a_n \mu^n$$

with

$$a_n = \int_{-\infty}^{\infty} e^{-2|x|} \, [s(x)]^n \, dx.$$

Since $|a_n| \leq s_0^n$, the integrated power series is certainly convergent for $|\mu| < \dfrac{1}{2s_0}$. Hence

$$\varepsilon = \mu + \sum_{n=1}^{\infty} a_n \mu^{n+1}, \quad \text{for} \quad |\mu| < \frac{1}{2s_0};$$

by the reversion theorem for power series,

$$\mu = \varepsilon + \varepsilon^2 \mu_1 + \cdots$$

is convergent for small $|\varepsilon|$. Therefore we find

$$\lambda = \frac{\varepsilon}{\mu} = 1 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots,$$

again convergent for small $|\varepsilon|$, which was the statement to be proved. Incidentally from $\lambda_1 = -\mu_1$ and

$$\varepsilon = (\varepsilon + \varepsilon^2\mu_1 + \cdots) + a_1 (\varepsilon + \mu_1\varepsilon^2 + \cdots)^2 + \cdots$$

we note that $\lambda_1 = -\mu_1 = a_1$, i.e.

$$\lambda_1 = \int_{-\infty}^{\infty} e^{-2|x|} \, s(x) \, dx$$

in agreement with our previous computation.

Thus we have shown: The equation

$$\int_{-\infty}^{\infty} e^{-|x|-|y|}\, u(y)\, dy + \varepsilon s(x)\, u(x) = \lambda u(x)$$

has a point eigenvalue $\lambda(\varepsilon)$ which is a regular analytic function of ε for small $|\varepsilon|$ and which tends to 1 as $\varepsilon \to 0$, provided $s(x)$ is a bounded (piecewise continuous) function in $-\infty < x < \infty$. The equation has no point eigenvalue if $s(x) = x$.

We must always bear in mind that smallness of the perturbation parameter ε is not sufficient to ensure that an eigenvalue problem $(A + \varepsilon B)\, u = \lambda u$ has a regular analytic point eigenvalue $\lambda = \lambda_0 + \varepsilon\lambda_1 + \cdots$ even though the unperturbed problem $Au = \lambda u$ possesses the eigenvalue λ_0.

Thus the following two questions arise:

(1) What criteria of "smallness" of a perturbation term εBu will in general guarantee the regular analytic dependence of point eigenvalues and corresponding eigenfunctions?

(In our example such criteria were the boundedness of $s(x)$ in $-\infty < x < \infty$ and the smallness of $|\varepsilon|$.)

(2) If is one such perturbation term the as $\varepsilon Bu = \varepsilon s(x)\, u$, what is the meaning of the "approximation" computed by formally applying the perturbation method?

The eigenvalue problem just dealt with exhibits the same difficulty as that which occurs in the wave equation for the *Stark effect*. E. Schrödinger † applied the first perturbation calculation used in quantum mechanics to this example. An electron with charge $-e$ moves in the electric field produced by a nucleus carrying the positive charge Ze. In addition, there is imposed an external electric field uniform in the direction of the x-axis. In suitable units the wave equation in this situation is

$$-(u_{xx} + u_{yy} + u_{zz}) - \frac{2Z}{r}\, u + \varepsilon x u = \lambda u,$$

where $r = (x^2 + y^2 + z^2)^{1/2}$ and ε is a measure of the strength of the external field. The positive integer Z and the real number ε are prescribed. A number λ_0 is said to be a point eigenvalue if for $\lambda = \lambda_0$ there is a (possibly)

† E. Schrödinger, Abhandlungen zur Wellenmechanik, 1926.

complex-valued solution $u \not\equiv 0$ of the differential equation such that

$$\iiint_{-\infty}^{\infty} |u(x, y, z)|^2 \, dx \, dy \, dz < \infty .$$

Any such solution $u(x, y, z)$ is called an eigenfunction belonging to λ_0. For $\varepsilon = 0$ all the point eigenvalues are known. One of these eigenvalues (the lowest in fact) is $\lambda_0 = -Z^2$ with a corresponding eigenfunction $u(x, y, z) = e^{-Zr}$. We have indeed

$$-\Delta u - \frac{2Z}{r} u = -\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \frac{2Z}{r}\right) e^{-Zr} = -Z^2 u .$$

"Normalizing" u we obtain the eigenfunction $\phi_0 = \frac{Z^{3/2}}{\pi^{1/2}} e^{-Zr}$.

If $\varepsilon \neq 0$ the problem is more complicated. We should like to find a solution of the form

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots, \qquad \lambda_0 = -Z^2$$

$$u = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots, \quad \phi_0 = \frac{Z^{3/2}}{\pi^{1/2}} e^{-Zr},$$

hoping that such expansions converge if $|\varepsilon|$ is very small. The computation of $\lambda_1, \phi_1, \ldots$ is now more complicated of course. The first order approximation $\lambda_0 + \varepsilon\lambda_1$ is immediately obtained, however, because

$$\lambda_1 = \iiint_{-\infty}^{\infty} x\phi_0^2 \, dx \, dy \, dz = 0,$$

for the integrand is an odd function of x. But it is again true that our eigenvalue problem has a purely continuous spectrum if $\varepsilon \neq 0$. There is no point eigenvalue $\lambda(\varepsilon)$ for $\varepsilon \neq 0$, and the meaning of the "approximation" $\lambda(\varepsilon) \approx \lambda_0 + \varepsilon\lambda_1$ is not at all clear.

The assertion that for $\varepsilon \neq 0$ there is a purely continuous spectrum is not easy to demonstrate. It amounts to proving:

If a complex-valued function $u(x, y, z) \not\equiv 0$ has continuous second derivatives and satisfies the equation

$$-\Delta u - \frac{2Z}{r} u + \varepsilon x u = \lambda u$$

for some real number $\varepsilon \neq 0$ and some complex number λ, then

$$\iiint_{-\infty}^{\infty} |u|^2 \, dx \, dy \, dz = \infty .$$

It would be desirable to give a direct proof of this statement.