Stratified Flows

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PREFACE TO SECOND EDITION

"Stratified Flows" is the name I have given to this second edition of "Dynamics of Nonhomogeneous Fluids" for several reasons: The new title is shorter, the term *stratified flows* is now well established in the literature, and above all, usage has endowed the term with an association with gravity effects, in which the most interesting features of the dynamics of nonhomogeneous fluids reside.

I have taken the opportunity of this second edition to make corrections, omit some parts that now seem unimportant, and add new sections to the book. I have also added after each chapter critical notes on new results (available after 1965) that are closely related to the topics discussed in the main body of the book. These notes are not exhaustive, and to make amends for their sparseness I have provided at the end of the book an extensive bibliography, which has been enlarged and brought up to date. I hope that this bibliography will be useful to anyone seeking to inform himself about more recent developments in the dynamics of stratified flows. In other respects I have preserved the structure and points of view of the first edition.

I express my thanks to Mrs. Beverly Pyle for ably typing the manuscript for the second edition. Two friends, Y. C. Fung of La Jolla and Milton van Dyke of Palo Alto, have given me great encouragement in the course of my work on this edition. Professor van Dyke's unique combination of generosity and enthusiasm was just the nourishment I needed to complete this rather arduous task. Much of the revision of this book was done in 1978 during my sojourn in Karlsruhe, which was made possible by a Humboldt award. To the Alexander von Humboldt Foundation I express my sincere appreciation for giving me the time for undisturbed work.

Finally, I thank the staff at Academic Press for their splendid cooperation.

PREFACE TO FIRST EDITION

This book deals with the flow of a fluid of variable density or entropy in a gravitational field. Without gravity, the heterogeneity of a fluid can have only minor effects on its behavior. Indeed, the totality of solutions for the motion of a heterogeneous fluid can be shown to be completely equivalent to the totality of solutions for a homogeneous fluid, provided the flow is steady and the fluid inviscid and nondiffusive. Without heterogeneity, gravity has no effect whatsoever on the kinematics or dynamics of the flow, aside from contributing to the total pressure a hydrostatic part. Of course, since a free surface marks the boundary between a fluid and another of negligible inertia and viscosity, its presence implies the presence of heterogeneity, and, in fact, heterogeneity in an extreme form. If heterogeneity and gravity are both present, the situation is not merely more complicated. Often their interplay produces striking phenomena entirely unexpected.

Since gravity is omnipresent, and fluid homogeneity an exception rather than a rule, the subject dealt with in this book is relevant to most flows occurring in nature. If gravity effects have been totally ignored by aerodynamicists, it is because the airplane is too small and too fast for gravity effects to be appreciable. The magnificent development of aerodynamics since the turn of the century need not obscure the fact that there remain wide areas of fluid mechanics, no less challenging and rewarding, awaiting the energy of the scientific worker for their exploration. It is hoped that this little book will provide an introduction to one of these areas.

It is quite impossible to be exhaustive in the treatment, and, since this book is not intended to be a handbook, exhaustiveness is perhaps not even desirable. I have tried to use gravity as the warp and heterogeneity as the woof of a mat underlying and unifying the material presented. On it a few interlocking patterns are discernible. There is the pattern of particular history and its utilization (Chapter 1, Section 2; Chapter 3, Sections 1–14; and

Chapter 5, Section 4). There is the use of singularities and the inverse method for dealing with large-amplitude motions (most of Chapter 3 and a good part of Chapter 5). There is the pattern of eigenvalue problems (Chapters 2 and 4). In Chapter 2, the threads are provided by Sturm, Liouville, and Bôcher, whereas much more modern fabric is used in Chapter 4. From the point of view of content, Chapter 1 gives some general results serving as preliminaries to the following three chapters, which might be of some interest to workers in meteorology, oceanography, and engineering. Chapter 5 deals exclusively with seepage flow in porous media; it is hoped that the many recent results presented there, though far from being exhaustive, may be useful to civil engineers dealing with ground water flow and chemical engineers interested in oil seepage in the ground. Chapters 2 and 4 would be more useful to the oceanographer and the meteorologist if Coriolis effects of the earth's rotation had been discussed more thoroughly. The exclusion of a thorough discussion of these effects is not merely a matter of economy of space. I am afraid that such a discussion would distract from the main point of view, and I content myself with the provision of a separate chapter (Chapter 6), in which the analogy between the flow of a heterogeneous fluid in a gravitational field and the flow of an accelerating or rotating fluid is presented. This chapter is therefore a little rug on which the patterns of the main rug are traced out, in the hope of giving some satisfying sense of unity between the two categories of fluid flow.

Many familiar results in the flow of a heterogeneous fluid in a gravitational field are missing in this book. The most obvious ones are those on water waves and those on gravitational convection. There seems to be little need and even less possibility of including the extensive results on water waves presented in Lamb's and Professor Stoker's excellent books. Much of the familiar results on gravitational convection not mentioned in this book can be found in books on heat transfer. Although the analogy of the flow of a conducting fluid in a magnetic field to that of a stratified or rotating fluid is sometimes mentioned in Chapter 6, for full information on hydromagnetic stability the reader must be referred to the literature, and especially to Professor S. Chandrasekhar's excellent and extensive book. I have not included any detailed information on turbulence in a stratified fluid because it seems to me that the time for such an inclusion has not yet arrived.

The bibliography, like the subject matter, is not exhaustive. But if the reader looks up the references given in the papers and books referred to in this book, and repeats the process, it is unlikely that he will miss many important papers written before 1963. I have not read all the papers listed in the bibliography, but I thought a fairly extensive list might be useful.

It is my pleasant duty to thank the many people who have directly or indirectly contributed to this book. All those whose names are mentioned in the text have contributed to my understanding of the subject. I owe my initial interest in gravity effects on fluid flow to Professor Hunter Rouse, in hydrodynamics to Professor John S. McNown, and in hydrodynamic stability to Professor Chia-Chiao Lin. Dr. George K. Batchelor encouraged the writing of this book when it was first conceived at Cambridge, England, in 1960. Through both his work and his personal encouragement, Sir Geoffrey Taylor has been a constant source of inspiration. I wish to express my appreciation to Professors Otto Laporte, Louis N. Howard, and John W. Miles, who kindly read the manuscript and gave many valuable comments and suggestions, and to Dr. Walter R. Debler, who often participated in my research work with enthusiasm. I have been much encouraged in this work by Professor Yuan-Cheng Fung, a friend since 1934, and by Professor Thomas Farrell, a teacher of good writing to many of his friends. Almost all of my own research work in the field covered by this book has been sponsored by the Army Research Office (Durham) and by the National Science Foundation, which awarded me a senior post-doctoral fellowship in 1959–60 and a research grant in 1961. To both I am grateful. I also wish to thank Mrs. Jane Lamb and Miss Joanna Zaparyniuk for their patient and skillful typing of the manuscript. Finally, it is a great pleasure to express my sincere appreciation to The Macmillan Company for its painstaking and excellent work in putting the book in print and to record my special indebtedness to Mr. A. H. McLeod of Macmillan, for his patient and wonderful cooperation.

GUIDES FOR THE READER

Equation numbers are consecutively numbered only within each chapter. If an equation and a reference to it are in the same chapter, its number is used without further identification. If an equation in one chapter is referred to in the text of another chapter, its number is prefixed by the number of the chapter in which it appears. Thus, if Eq. (18) in Chapter 1 is referred to in Chapter 3, it is referred to as (1.18).

In the Bibliography, the italic number following the name of a journal is the volume number. The numbers (or number) following the volume numbers of a journal are page numbers (or number).

In a book covering such a wide range of subjects, it is difficult to have a system of symbols without sacrificing either the one-to-one correspondence or the customary use of some of the familiar symbols. I have tried to preserve the usage of familiar symbols and to achieve as much consistency and one-to-one correspondence as possible. When one symbol is used to denote more than one quantity, care has been taken to ensure that confusion is not likely to arise. I often define a symbol immediately after it is introduced, whether or not it has been defined before. Thus some symbols are defined several times. In this way I hope to prevent the annoyance readers often experience when obliged to turn to a list of symbols, or to take on a long journey to locate the place where a particular symbol is first introduced—sometimes even when a list of symbols is given, because the terse explanation in that list does not suffice. A list of symbols is not provided because I believe it will not be needed.

CONTENTS

3. STEADY FLOWS OF FINITE AMPLITUDE

4. HYDRODYNAMIC STABILITY

Chapter 1

PRELIMINARIES

1. GENERAL DISCUSSION OF THE EFFECTS OF DENSITY OR ENTROPY VARIATION

For an inviscid* fluid, the equations of motion are the Euler equations

$$\rho\left(\frac{\partial u_i}{\partial t} + u_\alpha \frac{\partial u_i}{\partial x_\alpha}\right) = -\frac{\partial p}{\partial x_i} + \rho X_i, \tag{1}$$

in which t is the time, ρ is the density of the fluid, p is the pressure, X_i is the ith component of the body force per unit mass, u_i is the velocity component in the direction of x_i, with $i = 1$, 2, and 3, and x_1, x_2, and x_3 are Cartesian coordinates. In (1), repeated indices in one term indicate summation. Thus,

$$u_\alpha \frac{\partial u_i}{\partial x_\alpha} = u_1 \frac{\partial u_i}{\partial x_1} + u_2 \frac{\partial u_i}{\partial x_2} + u_3 \frac{\partial u_i}{\partial x_3}.$$

This convention will be used throughout the book unless otherwise stated.

An examination of (1) reveals immediately that a nonuniformity in density will have two effects for an incompressible fluid. First, since density is a measure of the resistance of the fluid to acceleration, a nonuniformity in density is a nonuniformity in inertia, which will affect the flow except in the trivial case of steady parallel motion. Second, whenever a body force is present a nonuniformity in density is necessarily accompanied by a nonuniformity in body force per unit volume, which in general will affect the flow. The most important body force on earth is of course the gravitational force. In most flows of a nonhomogeneous fluid the inertia effect and the gravity effect of density variation are both present, and their interplay is the most intriguing aspect of such flows.

* If the fluid is a gas, inviscidness implies that the volume viscosity as well as the ordinary viscosity is zero.

If the fluid is compressible, the matter must be scrutinized with more care. Density may change as a direct consequence of a change in pressure, or an indirect consequence of a change in speed. But if the change of state of all the fluid particles is sufficiently slow and heat conduction is neglected, the entropy of each of the particles will remain constant,* and for each particle there is a unique relationship between the pressure and the density,

$$p = f(\rho). \tag{2}$$

If the entropies of all the particles are the same, this relationship is the same for the *entire* field of flow, and the flow is called homentropic. As the density varies in a homentropic flow from place to place at the same instant, and (in unsteady flows) from time to time at the same point, the pressure varies according to (2), so that the integral dp/ρ is an exact differential and $\int dp/\rho$ exists. Then (1) can be written as

$$\frac{\partial u_i}{\partial t} + u_\alpha \frac{\partial u_i}{\partial x_\alpha} = -\frac{\partial}{\partial x_i}\int \frac{dp}{\rho} + X_i. \tag{3}$$

Thus the density ρ need not be associated with either the acceleration or the body force per unit mass, but can be absorbed in the term representing the pressure gradient, so that the *direct* inertia and gravity effects of density variation as discussed in the preceding paragraph are no longer manifest. This is not to say that the flow pattern will be unaffected by the density variation, because the equation of continuity

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_\alpha)}{\partial x_\alpha} = 0 \tag{4}$$

has to be satisfied together with (2) and (3), and for high enough speeds the effect of density variation can be very pronounced. Indeed, a major part of ordinary aerodynamics is devoted to a study of Eqs. (2), (3), and (4), with the term X_i neglected. For large-scale flows in the atmosphere, the body-force (or gravity) term must be retained. But so long as (2) holds for the whole field of flow, gravity affects the flow only through the density, which is dependent upon it. As will be shown later in this book, if the flow is nonhomentropic, gravity has, *in addition* to this effect through the density, a far-reaching effect entirely absent in homentropic flows.

For the flow of a compressible fluid, it is therefore more meaningful to consider, not the effects of density variation, but the effects of entropy variation. For a gas with a constant ratio (γ) of the specific heats c_p and c_v,

$$\frac{\rho}{p^{1/\gamma}} = \text{constant} \cdot e^{-S/c_p}, \tag{5}$$

* The flow is then called isentropic.

so that $\rho/p^{1/\gamma}$ is a function of the entropy S alone. The effects of entropy variation can be brought out vividly by dividing (1) throughout by $p^{1/\gamma}$. The result is

$$\frac{\rho}{p^{1/\gamma}}\left(\frac{\partial u_i}{\partial t} + u_\alpha \frac{\partial u_i}{\partial x_\alpha}\right) = -\frac{\partial}{\partial x_i}\left(\frac{\gamma}{\gamma - 1} p^{(\gamma-1)/\gamma}\right) + \frac{\rho}{p^{1/\gamma}} X_i. \tag{6}$$

Equation (6) as applied to a nonhomentropic gas can be compared with (1) as applied to a liquid of variable density. The quantity $\rho/p^{1/\gamma}$ in (6) corresponds to the density ρ in (1), and the effects of its nonhomogeneity in a gas are analogous to the effects of density variation in a liquid. We can thus speak of them as the inertia and gravity effects of entropy variation. In a gravitational field, nonhomentropy of a gas often has a far-reaching and pronounced effect on the flow, which is represented mathematically by the last term in (6).

2. THE INERTIA EFFECT OF DENSITY OR ENTROPY VARIATION

The components of acceleration of a fluid particle are given by

$$a_i = \frac{\partial u_i}{\partial t} + u_\alpha \frac{\partial u_i}{\partial x_\alpha}.$$

Since the temporal part $(\partial u_i/\partial t)$ is linear in u_i, whereas the convective part $(u_\alpha\, \partial u_i/\partial x_\alpha)$ is quadratic in the velocity components, there is no simple law to evaluate the effect of density (or entropy) variation on the velocity distribution. The discussion is profitable only if temporal and convective accelerations are discussed separately.

If the flow is unidirectional* (in the x_1-direction, say) so that the acceleration consists only of the temporal part, (1) can be written, with the last term neglected (because we are concerned at the moment with inertia effects of density variation),

$$\rho \frac{\partial u_1}{\partial t} = -\frac{\partial p}{\partial x_1}, \tag{7}$$

the other two equations contained in (1) being trivial, since $u_2 = u_3 = 0$. If the fluid is incompressible and, in addition, ρ is constant along the direction of flow,

$$\frac{\partial \rho}{\partial x_1} = 0. \tag{8}$$

Now the general equation of continuity is (4). The equation of incompressibility is, in general,

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + u_\alpha \frac{\partial \rho}{\partial x_\alpha} = 0, \tag{9}$$

* This situation can be realized by a wave maker in a confined channel at large distances from the wave maker. See Chapter 2, Section 15.

which states that the density of a fluid particle remains unchanged as the particle moves about. From (4) and (9) it follows that the equation of continuity for an incompressible fluid is

$$\frac{\partial u_\alpha}{\partial x_\alpha} = 0, \tag{10}$$

whether or not the flow is steady, and whether or not the fluid is homogeneous. In the special case discussed here, (10) becomes

$$\frac{\partial u_1}{\partial x_1} = 0, \tag{11}$$

and (9) becomes, by virtue of (8),

$$\frac{\partial \rho}{\partial t} = 0. \tag{12}$$

If ρ_0 is a reference density, and the new variables

$$u_1' = \frac{\rho}{\rho_0} u_1 \qquad \text{and} \qquad p' = p \tag{13}$$

are introduced, (12) permits (7) to be written as

$$\rho_0 \frac{\partial u_1'}{\partial t} = -\frac{\partial p'}{\partial x_1}, \tag{14}$$

and (8) permits (11) to be written as

$$\frac{\partial u_1'}{\partial x_1} = 0. \tag{15}$$

Thus (7) and (11) have been replaced by (14) and (15), which govern unidirectional flow of a homogeneous fluid of constant density ρ_0. Therefore, under the stated conditions, the effect of density variation on the velocity distribution is to make the velocity proportional to that of a homogeneous fluid subjected to comparable boundary conditions by the factor ρ_0/ρ. This is quite reasonable from the physical point of view.

Unfortunately, a transformation similar to that embodied in (13) cannot be found to reduce the equations governing unsteady flows of a nonhomentropic gas to those governing unsteady flows of a homentropic gas, even under the restriction of unidirectional flow and an equation like (8), with ρ replaced by $\rho p^{-1/\gamma}$.

If the flow is steady, so that the acceleration is purely convective, the inertia effect of density variation on the motion of an incompressible fluid can be evaluated by a rule different from but as simple as (13). With the body-force term neglected, (1) is now

$$\rho u_\alpha \frac{\partial u_i}{\partial x_\alpha} = -\frac{\partial p}{\partial x_i}, \tag{16}$$

The equation of continuity is still (10), but the equation of incompressibility now has the form

$$u_\alpha \frac{\partial \rho}{\partial x_\alpha} = 0. \tag{17}$$

If the new variables [Yih, 1958]

$$u_i' = \sqrt{\frac{\rho}{\rho_0}}\, u_i, \qquad p' = p \tag{18}$$

are introduced, (16) and (10) can be written, by virtue of (17), as

$$\rho_0 u_\alpha' \frac{\partial u_i'}{\partial x_\alpha} = -\frac{\partial p'}{\partial x_i}, \tag{19}$$

and

$$\frac{\partial u_\alpha'}{\partial x_\alpha} = 0. \tag{20}$$

But (19) and (20) govern the flow of a homogeneous fluid of constant density. Therefore it can be stated that, with gravity effects neglected, to every flow (called associated flow for convenience) of a homogeneous fluid of constant density, rotational or irrotational, correspond infinitely many stratified flows which are related to the associated flow (defined by u_i' and p') through equations (18). Thus, with gravity effect neglected, every known flow in classical hydrodynamics represents a class of stratified flows with the same flow pattern but different velocities.

A similar development for a compressible fluid with variable entropy is possible if its flow is steady, and if the change of state of each fluid particle is isentropic. Isentropy presumes a gradual change of state without heat conduction, and is expressed mathematically by the equation

$$\frac{DS}{Dt} = \frac{\partial S}{\partial t} + u_\alpha \frac{\partial S}{\partial x_\alpha} = 0, \tag{21}$$

in which S is the entropy. If the specific heats of an ideal gas are constant, $\rho/p^{1/\gamma}$ is a function of S alone, as stated in (5). Hence isentropy can also be expressed by

$$\left(\frac{\partial}{\partial t} + u_\alpha \frac{\partial}{\partial x_\alpha}\right)\left(\frac{\rho}{p^{1/\gamma}}\right) = 0. \tag{22}$$

For steady flows, the equation of isentropy is

$$u_\alpha \frac{\partial}{\partial x_\alpha}\left(\frac{\rho}{p^{1/\gamma}}\right) = 0. \tag{23}$$

Now the equations of motion are still (16), with the gravity term neglected. The equation of continuity is now, for steady flows,

$$\frac{\partial(\rho u_\alpha)}{\partial x_\alpha} = 0. \tag{24}$$