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A COURSE OF MODERN ANALYSIS

By

E. T. WHITTAKER

and

G. N. WATSON

A COURSE OF
MODERN ANALYSIS

AN INTRODUCTION TO THE GENERAL THEORY OF INFINITE PROCESSES AND OF ANALYTIC FUNCTIONS; WITH AN ACCOUNT OF THE PRINCIPAL TRANSCENDENTAL FUNCTIONS

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PREFACE

TO THE FOURTH EDITION

ADVANTAGE has been taken of the preparation of the fourth edition of this work to add a few additional references and to make a number of corrections of minor errors.

Our thanks are due to a number of our readers for pointing out errors and misprints, and in particular we are grateful to Mr E. T. Copson, Lecturer in Mathematics in the University of Edinburgh, for the trouble which he has taken in supplying us with a somewhat lengthy list.

E. T. W.
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[NOTE. The decimal system of paragraphing, introduced by Peano, is adopted in this work. The integral part of the decimal represents the number of the chapter and the fractional parts are arranged in each chapter in order of magnitude. Thus, e.g., on pp. 187, 188, § 9·632 precedes § 9·7 because $9{\cdot}632 < 9{\cdot}7$.]

PART I

THE PROCESSES OF ANALYSIS

CHAPTER I

COMPLEX NUMBERS

1·1. *Rational numbers.*

The idea of a set of numbers is derived in the first instance from the consideration of the set of *positive* integral numbers*, or *positive integers*; that is to say, the numbers 1, 2, 3, 4, ... Positive integers have many properties, which will be found in treatises on the Theory of Integral Numbers; but at a very early stage in the development of mathematics it was found that the operations of Subtraction and Division could only be performed among them subject to inconvenient restrictions; and consequently, in elementary Arithmetic, classes of numbers are constructed such that the operations of subtraction and division can always be performed among them.

To obtain a class of numbers among which the operation of subtraction can be performed without restraint we construct the class of *integers*, which consists of the class of positive† integers ($+1, +2, +3, \ldots$) and of the class of negative integers ($-1, -2, -3, \ldots$) and the number 0.

To obtain a class of numbers among which the operations both of subtraction and of division can be performed freely‡, we construct the class of *rational numbers*. Symbols which denote members of this class are $\frac{1}{2}$, 3, 0, $-\frac{15}{7}$.

We have thus introduced three classes of numbers, (i) the *signless integers*, (ii) the *integers*, (iii) the *rational numbers*.

It is not part of the scheme of this work to discuss the construction of the class of integers or the logical foundations of the theory of rational numbers§.

The extension of the idea of number, which has just been described, was not effected without some opposition from the more conservative mathematicians. In the latter half of the eighteenth century, Maseres (1731–1824) and Frend (1757–1841) published works on Algebra, Trigonometry, etc., in which the use of negative numbers was disallowed, although Descartes had used them unrestrictedly more than a hundred years before.

* Strictly speaking, a more appropriate epithet would be, not *positive*, but *signless*.

† In the strict sense.

‡ With the exception of division by the rational number 0.

§ Such a discussion, defining a rational number as an ordered number-pair of integers in a similar manner to that in which a complex number is defined in § 1·3 as an ordered number-pair of real numbers, will be found in Hobson's *Functions of a Real Variable*, §§ 1–12.

A rational number x may be represented to the eye in the following manner:

If, on a straight line, we take an origin O and a fixed segment OP_1 (P_1 being on the right of O), we can measure from O a length OP_x such that the ratio OP_x/OP_1 is equal to x; the point P_x is taken on the right or left of O according as the number x is positive or negative. We may regard either the *point* P_x or the *displacement* OP_x (which will be written $\overline{OP_x}$) as representing the number x.

All the rational numbers can thus be represented by points on the line, but the converse is not true. For if we measure off on the line a length OQ equal to the diagonal of a square of which OP_1 is one side, it can be proved that Q does not correspond to any rational number.

Points on the line which do not represent rational numbers may be said to represent irrational numbers; thus the point Q is said to represent the irrational number $\sqrt{2} = 1{\cdot}414213\ldots$. But while such an explanation of the existence of irrational numbers satisfied the mathematicians of the eighteenth century and may still be sufficient for those whose interest lies in the applications of mathematics rather than in the logical upbuilding of the theory, yet from the logical standpoint it is improper to introduce geometrical intuitions to supply deficiencies in arithmetical arguments; and it was shewn by Dedekind in 1858 that the theory of irrational numbers can be established on a purely arithmetical basis without any appeal to geometry.

1·2. *Dedekind's* theory of irrational numbers.*

The geometrical property of points on a line which suggested the starting point of the arithmetical theory of irrationals was that, if all points of a line are separated into two classes such that every point of the first class is on the right of every point of the second class, there exists one and only one point at which the line is thus severed.

Following up this idea, Dedekind considered rules by which a separation† or *section* of *all* rational numbers into two classes can be made, these classes (which will be called the L-class and the R-class, or the left class and the right class) being such that they possess the following properties:

(i) At least one member of each class exists.

(ii) Every member of the L-class is less than every member of the R-class.

It is obvious that such a section is made by any rational number x; and x is either the greatest number of the L-class or the least number of the

* The theory, though elaborated in 1858, was not published before the appearance of Dedekind's tract, *Stetigkeit und irrationale Zahlen*, Brunswick, 1872. Other theories are due to Weierstrass [see von Dantscher, *Die Weierstrass'sche Theorie der irrationalen Zahlen* (Leipzig, 1908)] and Cantor, *Math. Ann.* v. (1872), pp. 123–130.

† This procedure formed the basis of the treatment of irrational numbers by the Greek mathematicians in the sixth and fifth centuries B.C. The advance made by Dedekind consisted in observing that a purely *arithmetical* theory could be built up on it.

R-class. But sections can be made in which no rational number x plays this part. Thus, since there is no rational number* whose square is 2, it is easy to see that we may form a section in which the R-class consists of the positive rational numbers whose squares exceed 2, and the L-class consists of all other rational numbers.

Then this section is such that the R-class has no least member and the L-class has no greatest member; for, if x be any positive rational fraction, and $y = \frac{x(x^2+6)}{3x^2+2}$, then $y - x = \frac{2x(2-x^2)}{3x^2+2}$ and $y^2 - 2 = \frac{(x^2-2)^3}{(3x^2+2)^2}$, so x^2, y^2 and 2 are in order of magnitude; and therefore given any member x of the L-class, we can always find a greater member of the L-class, or given any member x' of the R-class, we can always find a smaller member of the R-class, such numbers being, for instance, y and y', where y' is the same function of x' as y of x.

If a section is made in which the R-class has a least member A_2, or if the L-class has a greatest member A_1, the section determines a *rational-real* number; which it is convenient to denote by the *same*† symbol A_2 or A_1.

If a section is made, such that the R-class has no least member and the L-class has no greatest member, *the section determines an irrational-real number*‡.

If x, y are real numbers (defined by sections) we say that x is greater than y if the L-class defining x contains at least two§ members of the R-class defining y.

Let α, β, ... be real numbers and let A_1, B_1, ... be any members of the corresponding L-classes while A_2, B_2, ... are any members of the corresponding R-classes. The classes of which A_1, A_2, ... are respectively members will be denoted by the symbols (A_1), (A_2), ...

Then the *sum* (written $\alpha + \beta$) of two real numbers α and β is defined as the real number (rational or irrational) which is determined by the L-class $(A_1 + B_1)$ and the R-class $(A_2 + B_2)$.

It is, of course, necessary to prove that these classes determine a section of the rational numbers. It is evident that $A_1 + B_1 < A_2 + B_2$ and that at least one member of each of the classes $(A_1 + B_1)$, $(A_2 + B_2)$ exists. It remains to prove that there is, at most, *one* rational

* For if p/q be such a number, this fraction being in its lowest terms, it may be seen that $(2q - p)/(p - q)$ is another such number, and $0 < p - q < q$, so that p/q is not in its lowest terms. The contradiction implies that such a rational number does not exist.

† This causes no confusion in practice.

‡ B. A. W. Russell defines the class of real numbers as *actually being* the class of all L-classes; the class of real numbers whose L-classes have a greatest member corresponds to the class of rational numbers, and though the rational-real number x which corresponds to a rational number x is conceptually distinct from it, no confusion arises from denoting both by the same symbol.

§ If the classes had only one member in common, that member might be the greatest member of the L-class of x and the least member of the R-class of y.

number which is greater than every A_1+B_1 and less than every A_2+B_2; suppose, if possible, that there are two, x and y $(y>x)$. Let a_1 be a member of (A_1) and let a_2 be a member of (A_2); and let N be the integer next greater than $(a_2-a_1)/\{\frac{1}{2}(y-x)\}$. Take the last of the numbers $a_1+\frac{m}{N}(a_2-a_1)$, (where $m=0, 1, \ldots N$), which belongs to (A_1) and the first of them which belongs to (A_2); let these two numbers be c_1, c_2. Then

$$c_2-c_1=\frac{1}{N}(a_2-a_1)<\tfrac{1}{2}(y-x).$$

Choose d_1, d_2 in a similar manner from the classes defining β; then

$$c_2+d_2-c_1-d_1<y-x.$$

But $c_2+d_2\geqslant y$, $c_1+d_1\leqslant x$, and therefore $c_2+d_2-c_1-d_1\geqslant y-x$; we have therefore arrived at a contradiction by supposing that two rational numbers x, y exist belonging neither to (A_1+B_1) nor to (A_2+B_2).

If every rational number belongs either to the class (A_1+B_1) or to the class (A_2+B_2), then the classes (A_1+B_1), (A_2+B_2) define an irrational number. If one rational number x exists belonging to neither class, then the L-class formed by x and (A_1+B_1) and the R-class (A_2+B_2) define the rational-real number x. In either case, the number defined is called the sum $\alpha+\beta$.

The difference $\alpha-\beta$ of two real numbers is defined by the L-class (A_1-B_2) and the R-class (A_2-B_1).

The product of two positive real numbers α, β is defined by the R-class (A_2B_2) and the L-class of all other rational numbers.

The reader will see without difficulty how to define the product of negative real numbers and the quotient of two real numbers; and further, it may be shewn that real numbers may be combined in accordance with the associative, distributive and commutative laws.

The aggregate of rational-real and irrational-real numbers is called the aggregate of real numbers; for brevity, rational-real numbers and irrational-real numbers are called rational and irrational numbers respectively.

1·3. *Complex numbers.*

We have seen that a real number may be visualised as a displacement along a definite straight line. If, however, P and Q are any two points in a plane, the displacement $\overline{PQ}$ needs two real numbers for its specification; for instance, the differences of the coordinates of P and Q referred to fixed rectangular axes. If the coordinates of P be (ξ, η) and those of Q $(\xi+x, \eta+y)$, the displacement $\overline{PQ}$ may be described by the symbol $[x, y]$. We are thus led to consider the association of real numbers in ordered* pairs. The natural definition of the sum of two displacements $[x, y]$, $[x', y']$ is the displacement which is the result of the successive applications of the two displacements; it is therefore convenient to define the sum of two number-pairs by the equation

$$[x, y]+[x', y']=[x+x', y+y'].$$

* The order of the two terms distinguishes the ordered number-pair $[x, y]$ from the ordered number-pair $[y, x]$.

The product of a number-pair and a real number x' is then naturally defined by the equation

$$x' \times [x, y] = [x'x, x'y].$$

We are at liberty to define the *product* of two number-pairs in any convenient manner; but the only definition, which does not give rise to results that are merely trivial, is that symbolised by the equation

$$[x, y] \times [x', y'] = [xx' - yy', xy' + x'y].$$

It is then evident that

$$[x, 0] \times [x', y'] = [xx', xy'] = x \times [x', y']$$

and

$$[0, y] \times [x', y'] = [-yy', x'y] = y \times [-y', x'].$$

The geometrical interpretation of these results is that the effect of multiplying by the displacement $[x, 0]$ is the same as that of multiplying by the real number x; but the effect of multiplying a displacement by $[0, y]$ is to multiply it by a real number y and turn it through a right angle.

It is convenient to denote the number-pair $[x, y]$ by the compound symbol $x + iy$; and a number-pair is now conveniently called (after Gauss) a *complex number*; in the fundamental operations of Arithmetic, the complex number $x + i0$ may be replaced by the real number x and, defining i to mean $0 + i1$, we have $i^2 = [0, 1] \times [0, 1] = [-1, 0]$; and so i^2 may be replaced by -1.

The reader will easily convince himself that the definitions of addition and multiplication of number-pairs have been so framed that we may perform the ordinary operations of algebra with complex numbers in exactly the same way as with real numbers, treating the symbol i as a number and replacing the product ii by -1 wherever it occurs.

Thus he will verify that, if a, b, c are complex numbers, we have

$$a + b = b + a,$$

$$ab = ba,$$

$$(a + b) + c = a + (b + c),$$

$$ab \,.\, c = a \,.\, bc,$$

$$a(b + c) = ab + ac,$$

and if ab is zero, then either a or b is zero.

It is found that algebraical operations, direct or inverse, when applied to complex numbers, do not suggest numbers of any fresh type; the complex number will therefore for our purposes be taken as the most general type of number.

The introduction of the complex number has led to many important developments in mathematics. Functions which, when real variables only are considered, appear as essentially distinct, are seen to be connected when complex variables are introduced:

thus the circular functions are found to be expressible in terms of exponential functions of a complex argument, by the equations

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix}), \quad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix}).$$

Again, many of the most important theorems of modern analysis are not true if the numbers concerned are restricted to be real; thus, the theorem that every algebraic equation of degree n has n roots is true in general only when regarded as a theorem concerning complex numbers.

Hamilton's quaternions furnish an example of a still further extension of the idea of number. A quaternion

$$w + xi + yj + zk$$

is formed from four real numbers w, x, y, z, and four number-units $1, i, j, k$, in the same way that the ordinary complex number $x + iy$ might be regarded as being formed from two real numbers x, y, and two number-units $1, i$. Quaternions however do not obey the commutative law of multiplication.

1·4. *The modulus of a complex number.*

Let $x + iy$ be a complex number, x and y being real numbers. Then the positive square root of $x^2 + y^2$ is called the *modulus* of $(x + iy)$, and is written

$$|x + iy|.$$

Let us consider the complex number which is the sum of two given complex numbers, $x + iy$ and $u + iv$. We have

$$(x + iy) + (u + iv) = (x + u) + i(y + v).$$

The modulus of the sum of the two numbers is therefore

$$\{(x + u)^2 + (y + v)^2\}^{\frac{1}{2}},$$

or

$$\{(x^2 + y^2) + (u^2 + v^2) + 2(xu + yv)\}^{\frac{1}{2}}.$$

But

$$\begin{aligned}\{|x + iy| + |u + iv|\}^2 &= \{(x^2 + y^2)^{\frac{1}{2}} + (u^2 + v^2)^{\frac{1}{2}}\}^2 \\ &= (x^2 + y^2) + (u^2 + v^2) + 2(x^2 + y^2)^{\frac{1}{2}}(u^2 + v^2)^{\frac{1}{2}} \\ &= (x^2 + y^2) + (u^2 + v^2) + 2\{(xu + yv)^2 + (xv - yu)^2\}^{\frac{1}{2}},\end{aligned}$$

and this latter expression is greater than (or at least equal to)

$$(x^2 + y^2) + (u^2 + v^2) + 2(xu + yv).$$

We have therefore

$$|x + iy| + |u + iv| \geqslant |(x + iy) + (u + iv)|,$$

i.e. *the modulus of the sum of two complex numbers cannot be greater than the sum of their moduli*; and it follows by induction that the modulus of the sum of any number of complex numbers cannot be greater than the sum of their moduli.

Let us consider next the complex number which is the product of two given complex numbers, $x+iy$ and $u+iv$; we have

$$(x+iy)(u+iv)=(xu-yv)+i(xv+yu),$$

and so

$$\begin{aligned}|(x+iy)(u+iv)| &= \{(xu-yv)^2+(xv+yu)^2\}^{\frac{1}{2}}\\ &= \{(x^2+y^2)(u^2+v^2)\}^{\frac{1}{2}}\\ &= |x+iy|\,|u+iv|.\end{aligned}$$

The modulus of the product of two complex numbers (and hence, by induction, of any number of complex numbers) *is therefore equal to the product of their moduli.*

1·5. *The Argand diagram.*

We have seen that complex numbers may be represented in a geometrical diagram by taking rectangular axes Ox, Oy in a plane. Then a point P whose coordinates referred to these axes are x, y may be regarded as representing the complex number $x+iy$. In this way, to every point of the plane there corresponds some one complex number; and, conversely, to every possible complex number there corresponds one, and only one, point of the plane. The complex number $x+iy$ may be denoted by a single letter* z. The point P is then called the *representative point* of the number z; we shall also speak of the number z as being the *affix* of the point P.

If we denote $(x^2+y^2)^{\frac{1}{2}}$ by r and choose θ so that $r\cos\theta=x$, $r\sin\theta=y$, then r and θ are clearly the radius vector and vectorial angle of the point P, referred to the origin O and axis Ox.

The representation of complex numbers thus afforded is often called the *Argand diagram*†.

By the definition already given, it is evident that r is the modulus of z. The angle θ is called the *argument*, or *amplitude*, or *phase*, of z.

We write $$\theta=\arg z.$$

From geometrical considerations, it appears that (although the modulus of a complex number is unique) the argument is not unique‡; if θ be a value of the argument, the other values of the argument are comprised in the expression $2n\pi+\theta$ where n is any integer, not zero. The *principal* value of $\arg z$ is that which satisfies the inequality $-\pi<\arg z\leqq\pi$.

* It is convenient to call x and y the *real* and *imaginary* parts of z respectively. We frequently write $x=R(z)$, $y=I(z)$.

† It was published by J. R. Argand, *Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques* (1806); it had however previously been used by Gauss, in his Helmstedt dissertation, 1799 (*Werke*, III. pp. 20–23), who had discovered it in Oct. 1797 (*Math. Ann.* LVII. p. 18); and Caspar Wessel had discussed it in a memoir presented to the Danish Academy in 1797 and published by that Society in 1798–9. The phrase *complex number* first occurs in 1831, Gauss, *Werke*, II. p. 102.

‡ See the *Appendix*, § A·521.

If P_1 and P_2 are the representative points corresponding to values z_1 and z_2 respectively of z, then the point which represents the value z_1+z_2 is clearly the terminus of a line drawn from P_1, equal and parallel to that which joins the origin to P_2.

To find the point which represents the complex number $z_1 z_2$, where z_1 and z_2 are two given complex numbers, we notice that if

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1),$$
$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

then, by multiplication,

$$z_1 z_2 = r_1 r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}.$$

The point which represents the number $z_1 z_2$ has therefore a radius vector measured by the product of the radii vectores of P_1 and P_2, and a vectorial angle equal to the sum of the vectorial angles of P_1 and P_2.

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MISCELLANEOUS EXAMPLES.

1. Shew that the representative points of the complex numbers $1+4i$, $2+7i$, $3+10i$, are collinear.

2. Shew that a parabola can be drawn to pass through the representative points of the complex numbers

$$2+i,\quad 4+4i,\quad 6+9i,\quad 8+16i,\quad 10+25i.$$

3. Determine the nth roots of unity by aid of the Argand diagram; and shew that the number of primitive roots (roots the powers of each of which give all the roots) is the number of integers (including unity) less than n and prime to it.

Prove that if $\theta_1, \theta_2, \theta_3, \ldots$ be the arguments of the primitive roots, $\Sigma\cos p\theta = 0$ when p is a positive integer less than $\frac{n}{abc\ldots k}$, where $a, b, c, \ldots k$ are the different constituent primes of n; and that, when $p = \frac{n}{abc\ldots k}$, $\Sigma\cos p\theta = \frac{(-)^\mu n}{abc\ldots k}$, where μ is the number of the constituent primes. (Math. Trip. 1895.)

CHAPTER II

THE THEORY OF CONVERGENCE

2·1. *The definition* of the limit of a sequence.*

Let $z_1, z_2, z_3, \ldots$ be an unending sequence of numbers, real or complex. Then, if a number l exists such that, corresponding to every positive† number ϵ, no matter how small, a number n_0 can be found, such that

$$|z_n - l| < \epsilon$$

for all values of n greater than n_0, *the sequence* (z_n) *is said to tend to the limit* l *as* n *tends to infinity.*

Symbolic forms of the statement‡ 'the limit of the sequence (z_n), as n tends to infinity, is l' are:

$$\lim_{n\to\infty} z_n = l, \quad \lim z_n = l, \quad z_n \to l \text{ as } n \to \infty.$$

If the sequence be such that, given an arbitrary number N (no matter how large), we can find n_0 such that $|z_n| > N$ for all values of n greater than n_0, we say that '$|z_n|$ tends to infinity as n tends to infinity,' and we write

$$|z_n| \to \infty.$$

In the corresponding case when $-x_n > N$ when $n > n_0$ we say that $x_n \to -\infty$.

If a sequence of real numbers does not tend to a limit or to ∞ or to $-\infty$, the sequence is said to *oscillate.*

2·11. *Definition of the phrase 'of the order of.'*

If (ζ_n) and (z_n) are two sequences such that a number n_0 exists such that $|(\zeta_n/z_n)| < K$ whenever $n > n_0$, where K is *independent of* n, we say that ζ_n is 'of the order of' z_n, and we write§

$$\zeta_n = O(z_n);$$

thus

$$\frac{15n+19}{1+n^3} = O\left(\frac{1}{n^2}\right).$$

If $\lim(\zeta_n/z_n) = 0$, we write $\zeta_n = o(z_n)$.

* A definition equivalent to this was first given by John Wallis in 1655. [*Opera*, I. (1695), p. 382.]

† The number zero is excluded from the class of positive numbers.

‡ The arrow notation is due to Leathem, *Camb. Math. Tracts*, No. 1.

§ This notation is due to Bachmann, *Zahlentheorie* (1894), p. 401, and Landau, *Primzahlen*, I. (1909), p. 61.

2·2. *The limit of an increasing sequence.*

Let (x_n) be a sequence of real numbers such that $x_{n+1} \geqslant x_n$ for all values of n; then *the sequence tends to a limit or else tends to infinity* (and so it does not oscillate).

Let x be any rational-real number; then either:

(i) $x_n \geqslant x$ for all values of n greater than some number n_0 depending on the value of x.

Or (ii) $x_n < x$ for every value of n.

If (ii) is not the case for *any* value of x (no matter how large), then $x_n \rightarrow \infty$.

But if values of x exist for which (ii) holds, we can divide the rational numbers into two classes, the L-class consisting of those rational numbers x for which (i) holds and the R-class of those rational numbers x for which (ii) holds. This section defines a real number α, rational or irrational.

And if ϵ be an arbitrary positive number, $\alpha - \frac{1}{2}\epsilon$ belongs to the L-class which defines α, and so we can find n_1 such that $x_n \geqslant \alpha - \frac{1}{2}\epsilon$ whenever $n > n_1$; and $\alpha + \frac{1}{2}\epsilon$ is a member of the R-class and so $x_n < \alpha + \frac{1}{2}\epsilon$. Therefore, whenever $n > n_1$,

$$|\alpha - x_n| < \epsilon.$$

Therefore $x_n \rightarrow \alpha$.

Corollary. A decreasing sequence tends to a limit or to $-\infty$.

Example 1. If $\lim z_m = l$, $\lim z_m' = l'$, then $\lim (z_m + z_m') = l + l'$.

For, given ϵ, we can find n and n' such that

(i) when $m > n$, $|z_m - l| < \frac{1}{2}\epsilon$, (ii) when $m > n'$, $|z_m' - l'| < \frac{1}{2}\epsilon$.

Let n_1 be the greater of n and n'; then, when $m > n_1$,

$$|(z_m + z_m') - (l + l')| \leqslant |(z_m - l)| + |(z_m' - l')|,$$
$$< \epsilon;$$

and this is the condition that $\lim (z_m + z_m') = l + l'$.

Example 2. Prove similarly that $\lim (z_m - z_m') = l - l'$, $\lim (z_m z_m') = ll'$, and, if $l' \neq 0$, $\lim (z_m / z_m') = l/l'$.

Example 3. If $0 < x < 1$, $x^n \rightarrow 0$.

For if $x = (1 + a)^{-1}$, $a > 0$ and

$$0 < x^n = \frac{1}{(1+a)^n} < \frac{1}{1+na},$$

by the binomial theorem for a positive integral index. And it is obvious that, given a positive number ϵ, we can choose n_0 such that $(1 + na)^{-1} < \epsilon$ when $n > n_0$; and so $x^n \rightarrow 0$.

2·21. *Limit-points and the Bolzano-Weierstrass* theorem.*

Let (x_n) be a sequence of real numbers. If any number G exists such

* This theorem, frequently ascribed to Weierstrass, was proved by Bolzano, *Abh. der k. böhmischen Ges. der Wiss.* v. (1817). [Reprinted in *Klassiker der Exakten Wiss.*, No. 153.] It seems to have been known to Cauchy.

that, for every positive value of ϵ, no matter how small, an unlimited number of terms of the sequence can be found such that

$$G - \epsilon < x_n < G + \epsilon,$$

then G is called a *limit-point*, or *cluster-point*, of the sequence.

Bolzano's theorem is that, *if $\lambda \leqslant x_n \leqslant \rho$, where λ, ρ are independent of n, then the sequence (x_n) has at least one limit-point.*

To prove the theorem, choose a section in which (i) the R-class consists of all the rational numbers which are such that, if A be any one of them, there are only a limited number of terms x_n satisfying $x_n > A$; and (ii) the L-class is such that there are an unlimited number of terms x_n such that $x_n \geqslant a$ for all members a of the L-class.

This section defines a real number G; and, if ϵ be an arbitrary positive number, $G - \frac{1}{2}\epsilon$ and $G + \frac{1}{2}\epsilon$ are members of the L and R classes respectively, and so there are an unlimited number of terms of the sequence satisfying

$$G - \epsilon < G - \tfrac{1}{2}\epsilon \leqslant x_n \leqslant G + \tfrac{1}{2}\epsilon < G + \epsilon,$$

and so G satisfies the condition that it should be a limit-point.

2·211. *Definition of 'the greatest of the limits.'*

The number G obtained in § 2·21 is called 'the greatest of the limits of the sequence (x_n).' The sequence (x_n) cannot have a limit-point greater than G; for if G' were such a limit-point, and $\epsilon = \frac{1}{2}(G' - G)$, $G' - \epsilon$ is a member of the R-class defining G, so that there are only a limited number of terms of the sequence which satisfy $x_n > G' - \epsilon$. This condition is inconsistent with G' being a limit-point. We write

$$G = \overline{\lim_{n\to\infty}}\, x_n.$$

The 'least of the limits,' L, of the sequence (written $\underline{\lim}_{n\to\infty}\, x_n$) is defined to be

$$-\overline{\lim_{n\to\infty}}\,(-x_n).$$

2·22. CAUCHY'S* THEOREM ON THE NECESSARY AND SUFFICIENT CONDITION FOR THE EXISTENCE OF A LIMIT.

We shall now shew that the necessary and sufficient condition for the existence of a limiting value of a sequence of numbers $z_1, z_2, z_3, \ldots$ is that, *corresponding to any given positive number ϵ, however small, it shall be possible to find a number n such that*

$$|z_{n+p} - z_n| < \epsilon$$

for all positive integral values of p. This result is one of the most important and fundamental theorems of analysis. It is sometimes called the *Principle of Convergence.*

* *Analyse Algébrique* (1821), p. 125.

First, we have to shew that this condition is *necessary*, i.e. that it is satisfied whenever a limit exists. Suppose then that a limit l exists; then (§ 2·1) corresponding to any positive number ϵ, however small, an integer n can be chosen such that

$$|z_n - l| < \tfrac{1}{2}\epsilon, \quad |z_{n+p} - l| < \tfrac{1}{2}\epsilon,$$

for all positive values of p; therefore

$$\begin{aligned} |z_{n+p} - z_n| &= |(z_{n+p} - l) - (z_n - l)| \\ &\leqslant |z_{n+p} - l| + |z_n - l| < \epsilon, \end{aligned}$$

which shews the *necessity* of the condition

$$|z_{n+p} - z_n| < \epsilon,$$

and thus establishes the first half of the theorem.

Secondly, we have to prove* that this condition is *sufficient*, i.e. that if it is satisfied, then a limit exists.

(I) Suppose that the sequence of *real* numbers (x_n) satisfies Cauchy's condition; that is to say that, corresponding to any positive number ϵ, an integer n can be chosen such that

$$|x_{n+p} - x_n| < \epsilon$$

for all positive integral values of p.

Let the value of n, corresponding to the value 1 of ϵ, be m.

Let λ_1, ρ_1 be the least and greatest of $x_1, x_2, \ldots x_m$; then

$$\lambda_1 - 1 < x_n < \rho_1 + 1,$$

for all values of n; write $\lambda_1 - 1 = \lambda$, $\rho_1 + 1 = \rho$.

Then, for all values of n, $\lambda < x_n < \rho$. *Therefore by the theorem of* § 2·21, *the sequence* (x_n) *has at least one limit-point* G.

Further, there cannot be more than one limit-point; for if there were two, G and H $(H < G)$, take $\epsilon < \frac{1}{4}(G - H)$. Then, by hypothesis, a number n exists such that $|x_{n+p} - x_n| < \epsilon$ for every positive value of p. But since G and H are limit-points, positive numbers q and r exist such that

$$|G - x_{n+q}| < \epsilon, \quad |H - x_{n+r}| < \epsilon.$$

Then $\quad |G - x_{n+q}| + |x_{n+q} - x_n| + |x_n - x_{n+r}| + |x_{n+r} - H| < 4\epsilon.$

But, by § 1·4, the sum on the left is greater than or equal to $|G - H|$.

Therefore $G - H < 4\epsilon$, which is contrary to hypothesis; so there is only one limit-point. Hence there are only a finite number of terms of the sequence outside the interval $(G - \delta, G + \delta)$, where δ is an arbitrary positive number;

* This proof is given by Stolz and Gmeiner, *Theoretische Arithmetik*, II. (1902), p. 144.