APPROXIMATE SOLUTION METHODS IN ENGINEERING MECHANICS

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Arthur P. Boresi[1] and Ken P. Chong[2]

[1] *Professor and Head, Department of Civil Engineering, University of Wyoming*

[2] *Director of Structures and Building Systems, National Science Foundation*

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Preface

The widespread use of digital computers has had a profound effect in engineering and science. On the one hand, it has resulted in many benefits. On the other hand, because of inadequate training and experience of the user, it has often led to the 'garbage in, garbage out' and the 'black box' syndromes. For example, with computers and appropriate software, we can model and analyze complex physical systems and problems. However, the efficient and accurate use of numerical results obtained from computer programs requires considerable background and advanced working knowledge to avoid blunders and the blind acceptance of computer results. In this book, we attempt to provide some of the background and knowledge necessary to avoid these pitfalls. In particular, we consider several of the most commonly used approximate methods employed in the solution of physical problems.

A realistic and successful solution of an engineering problem begins usually with an accurate physical model of the problem, and a proper understanding of the assumptions employed. In turn, this physical model is transformed into a mathematical model or problem. The solution of the mathematical problem is usually obtained by numerical methods that by definition are approximate. Except for relatively simple cases, the successful use of numerical methods depends heavily upon the use of digital computers, ranging in power from microcomputers to supercomputers depending upon the complexity of the problem.

Powerful approximation (numerical) methods are presented in this book. These methods include:

- weighted residuals methods
- finite difference methods
- finite element methods
- finite strip/layer/prism methods

The mathematical formulation of these methods is perfectly general. However, in this book, the applications deal mainly with the field of solid mechanics. An extensive list of references is provided.

Practicing engineers and scientists should find this book very readable and useful. Students may use the book both as a reference and as a text. This book is a sequel to an earlier book by the same authors (*Elasticity in Engineering Mechanics*, published by Elsevier Science Publishing Co., Inc., New York, 1987).

Arthur P. Boresi and Ken P. Chong

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Chapter 1

The Role of Approximate Solution Methods in Engineering

1.1 INTRODUCTION

The successful solution of a complex engineering problem begins with an accurate physical model of the problem. In turn, this physical model is transformed into a mathematical model. The solution of the mathematical model is usually obtained by numerical methods that are by definition approximate. The success of these numerical techniques rest in turn upon high-speed digital computers.

The finite difference method (FDM) (Mitchell and Griffiths, 1980; Anderson *et al.*, 1984) and the finite element method (FEM) (Bathe, 1982; Hughes, 1987; Cook *et al.*, 1989; Zienkiewicz and Taylor, 1989) are widely used numerical techniques. These methods are classified as 'domain' methods, in that the engineering system is analyzed either in terms of discretized finite grids (FDM) or finite elements (FEM) throughout the entire region of the system (body). Another method that has emerged as a powerful tool is the boundary element method (BEM) (Rizzo, 1967; Brebbia, 1978; Brebbia, 1984a, Cruse, 1988). In certain problems, this method has some distinct advantages over FDM and FEM for several reasons. In particular, a discretization of *only* the boundary of the domain of interest is necessary for BEM, hence, the name boundary element method.

All three of the above methods, and a variety of other specialized techniques, provide powerful means of treating complex boundary value problems of engineering. In a particular case, depending upon requirements, one of these methods may be more efficient in generating a solution than the others. For example, it may be more advantageous to use BEM for certain classes of linear problems characterized by infinite or semi-infinite domains, stress concentrations, three-dimensional structural effects, and so on (Beskos, 1989; Brebbia, 1984b).

A fourth method, the finite strip method (FSM) (and the associated finite layer method (FLM) and finite prism method (FPM)) (Cheung, 1976) falls somewhere between domain methods (FDM and FEM) and boundary methods (BEM), in that it reduces the dimensions of the problem before discretization of the domain. Other hybrid methods that combine advantages of several formulations have been proposed. For example, Golley and Grice (1989), Golley *et al.* (1987), and Petrolito *et al.* (1989) combined FEM and FSM to study plate bending problems.

In this book, we consider finite difference methods (Chapter 3), finite element methods (Chapter 4) and finite strip (finite layer and finite prism) methods (Chapter 5). In Chapter 2, we consider the fundamentals of approximate methods of analysis, including boundary solutions in terms of weighted residual methods (WRM). The different approximation techniques (FDM, FEM, FSM, BEM, etc.) may be represented as special cases of weighted residual formulations. They can be studied using the concepts of approximation and weighting that are fundamental to WRM and that are popular with engineers and mathematicians.

1.2 FIELDS OF APPLICATION

Approximation methods are employed generally in all fields of engineering, mathematics and science. For example, textbooks have been written dealing with applications in specialized subjects such as elasticity, plasticity, porous media flow, structural mechanics, fluid mechanics, aerodynamics, and so on (Boresi and Chong, 1987; Lubliner, 1990; Anderson *et al.*, 1984; Holzer, 1985). In this book, we are concerned mainly with formulations of WRM, FDM, FEM, and FSM, and with recent developments in computer implementations of these methods (Nelson, 1989). Selected references are given, with emphasis on engineering applications in the solid mechanics area.

1.3 FUTURE PROGRESS

We stand at the threshold of an explosive expansion of computer applications in numerical solutions of problems in all areas of engineering (Anderson *et al.*, 1984; Nelson, 1989; Beskos, 1989).

Major advances in finite element model technology have been outlined by Belytschko (1989). Numerical methods for solution of finite element systems are described by Wilson (1989). In addition, a broad range of topics on the application of modern computers to the solution of structural engineering problems is discussed in the proceedings of the 1989 Structures Congress (Nelson, 1989). These topics include artificial intelligence, parallel processing, optimization, knowledge-based systems, and computer-aided analysis and design. The present status and future developments in boundary element methods in structural analysis are discussed by Beskos (1989). Finally, it is anticipated that future work in hybrid methods will play a greater role in approximation methods (Golley and Grice, 1989; Costabel and Stephan, 1988).

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Chapter 2

Approximate Analysis; Weighted Residuals

2.1 INTRODUCTION

Exact analytical solutions to certain engineering boundary-value problems exist (Boresi and Chong, 1987). However, in many cases, the boundary-value problems of engineering cannot be solved exactly by currently available analytical methods. In such cases, approximate solutions are sought.

In some situations, we may be able to suitably simplify the physical model and obtain exact solutions to the modified problem. In other circumstances it may be advantageous to employ a direct attack on the equations through finite difference methods (Chapter 3) or by piecewise polynomial methods (such as finite element methods, Chapter 4). Alternatively, in some problems we may employ the classical method of separation of variables in conjunction with series methods (Chapter 5).

There exists a broad area of mathematics known as approximation theory (Shisha, 1968). The term is usually reserved for that branch of mathematics devoted to the approximation of general functions by means of simple functions. For example, in practice we may wish to approximate a real arbitrary function $f(x)$ by means of a polynomial $p(x)$ in some finite interval of space, say, $a \leq x \leq b$. The motivation for such an approximation is often one of simplification, particularly when the function $f(x)$ is too complicated to manipulate. Because the approximation, say $F(x)$, ordinarily differs from $f(x)$, questions immediately arise as to the manner in which we should proceed. As noted by Shisha, approximation theory considers such problems as the following:

(1) What kinds of functions $F(x)$ should we consider for ap-

proximating $f(x)$? In other words, the question of 'function form' or 'trial function' for the approximation is considered.

(2) How do we measure the 'accuracy' or 'goodness' of the approximation? To answer this question, we must establish some measure or 'norm' of accuracy or goodness of approximation.

(3) Of all possible forms or trial functions is there one that approximates $f(x)$ well? How well? Is there, by some standard of measure (norm), one approximation that fits $f(x)$ best? That is, for a given norm is there some 'best' approximation of $f(x)$? And if there is a best approximation, is it unique?

(4) How can we get approximations to $f(x)$ in practice?

The problem of obtaining approximate solutions to initial-value or boundary-value problems differs from that of obtaining the best approximation of known functions in that the exact solution—call it $y(x)$—is unknown. In addition, the solution is required to satisfy a differential equation as well as initial values and/or boundary values. Hence, although we wish to approximate $y(x)$ by some trial function $Y(x)$ in the 'best' possible sense, our problem is greatly complicated. Nevertheless, the ideas and concepts of general approximation theory can be employed in approximation solutions of initial-value and/or boundary-value problems of engineering.

In this chapter we briefly discuss concepts common to a broad class of approximation methods. In particular, we treat the *method of weighted residuals,* the most general of trial function methods (Crandall, 1956; Collatz, 1960; Finlayson, 1972). Depending upon the norm employed, we may show that the method of weighted residuals leads to well-known approximation methods (for example, the Galerkin method, collocation, least squares and so on). In other words, the method of weighted residuals unifies many approximation methods that are currently in use (Finlayson and Scriven, 1966; Reddy, 1986; Hughes, 1987; Cook, *et al.*, 1989). For example, we will see that the Rayleigh–Ritz method (Langhaar, 1989) is a weighted residual method with a particular choice of weighting function.

In Chapter 3 we treat in detail approximate solutions of engineering problems by finite difference methods. In Chapter 4, we develop finite element methods and apply them to two- and three-dimensional problems. In Chapter 5, we formulate finite strip, finite layer and finite prism methods for special applications.

2.2 THE APPROXIMATION PROBLEM (TRIAL FUNCTIONS; NORMS OR MEASURES OF ERROR)

We are here concerned primarily with approximate methods of solving boundary-value problems in engineering. In boundary-value problems in engineering, we are generally faced with determining a solution to a differential equation (or a system of differential equations) in a region R. The solution is required to meet certain conditions on the boundary B of the region. In many cases the given equation or equations do not possess a known exact solution. Accordingly, unless we are able to obtain the unknown exact solution, we are forced to find an approximation of the exact solution. The form of this approximation is often cast in terms of a *trial solution* or *trial function* $F(x; A)$, which is assumed to be compatible with the exact solution $E(x)$, where x is defined in a compact subset (region) of space. For example, x may denote n real variables, or it may denote three spatial coordinates in the domain R; in one-dimensional problems x denotes a single real variable. The symbol A stands for a collection of parameters $a_1, a_2, a_3, \ldots, a_n$. Thus $A = A:(a_1, a_2, a_3, \ldots, a_n)$. Unfortunately, *there is no general scientific method of determining which of the unlimited number of approximating functions (trial functions or forms) ordinarily available will lead to the most 'efficient' approximation of* $E(x)$. In practice, the choice of trial functions is often made on the basis of experience or intuition. For example, one may sense intuitively that a certain form of trial function (say, a polynomial or a Fourier series) may be suitable, but not have any method at hand to actually determine the required approximation.

The choice of a method of estimating the accuracy of the approximation (choice of norm) ordinarily is less important than that of trial function (choice of form). Generally speaking, if $F(x; A)$ is compatible with $E(x)$, then almost any reasonable norm will lead to an efficient approximation of $E(x)$. However, if $F(x; A)$ is not compatible with $E(x)$, an efficient approximation will not ordinarily be attained, regardless of the norm employed. Of course, the choice of the norm may affect the complexity of estimating the accuracy of the approximation. In some cases this factor may dictate the choice of norm. In other words, once the form or family of trial functions for the approximation is selected, we wish to select the best approximation possible within the family. Then the norms on which this best approximation rests are essentially unlimited in number. To simplify the choice somewhat, we will restrict the discussion that follows to two rather broadly employed methods of error measurement: the method of weighted residuals and the variational method. In either case, for linear boundary-value problems, the method leads to consideration of the solution of a set of simultaneous linear algebraic equations.

In Section 2.3 we consider the method of weighted residuals as applied to ordinary differential equations, and in Section 2.4, to partial differential equations. In Section 2.5 we outline briefly the variation method and show its relation to the method of weighted residuals.

2.3 METHOD OF WEIGHTED RESIDUALS (ORDINARY DIFFERENTIAL EQUATIONS)

Preliminary Remarks

In certain simplified situations the boundary-value problem may be reduced to one of ordinary differential equations in a single dependent variable. (See, for example, the axially symmetric and the spherically symmetric problems of elasticity; Boresi and Chong, 1987. See also the finite strip method of Section 5.1.) Furthermore, the method of separation of variables leads to ordinary differential equations (see Sections 6.9, 7.10 and 7.15, Boresi and Chong, 1987). Consequently, in this section we consider a method of evaluation of ordinary differential equations based upon the method of weighted residuals. First, we outline the general approximation method for ordinary differential equations.

As noted in Section 2.2, an approximate solution to differential equations is often sought by assuming that the exact solution $E(x)$ may be approximated by an expression of the form $F(x; a_1, a_2, a_3, \ldots, a_n)$, where $a_1, a_2, a_3, \ldots, a_n$ are arbitrary parameters to be chosen to best fit the exact solution $E(x)$. This best fit is related to a particular measure (norm) of the approximation. In selecting the form $F(x; a_1, a_2, a_3, \ldots, a_n)$, we have certain options available. For example, let $R + B = D$ be the domain of the boundary-value problem, where R is the interior of D bounded by surface B. Then we may choose F in one of the following ways (Collatz, 1960):

(1) The differential equation is satisfied exactly in R, and the a_i are selected to make F fit the boundary conditions on B in some best sense (norm). This method of selecting the a_i is called *the boundary method.*

(2) The boundary conditions are satisfied exactly on B, and the a_i are selected so that F satisfies the differential equation in the interior R in some best sense (norm). This method is called *the interior method*.

(3) The differential equation is not satisfied in R, nor are the boundary conditions satisfied on B. The a_i are chosen to satisfy the differential equation in R and the boundary conditions on B in some best sense. This method of determining the a_i is called a *boundary-interior method* or simply a *mixed method*.

In ordinary differential equations, interior methods are most often used, as even if we know the general solution to the differential equation (boundary method), we still must solve a set of n algebraic (linear or possibly nonlinear) equations in the a_i to satisfy the boundary conditions. In the boundary-value problem of ordinary differential equations, the boundary conditions are usually specified at two points of the region, say, $x = 0$ and $x = L$. Thus, we speak of the two-point boundary-value problem.

In boundary-value problems of partial differential equations, both boundary and interior methods are used. However, in many cases boundary methods are preferred, as satisfaction of boundary conditions, as far as integration is concerned, requires the evaluation of integrals over the boundary B rather than evaluation of integrals through the interior region R, as do interior methods. When the differential equations and boundary conditions are very complicated, some simplification may be possible with the use of mixed methods.

Method of Weighted Residuals

The method of weighted residuals seeks to produce a best approximate solution to a differential equation (subject to boundary conditions) through the use of trial functions. This use is also a feature of variational methods (See Section 2.5; Chapter 4; and Langhaar, 1989). Special widely used cases of the method of weighted residuals include the Galerkin method, the method of collocation, and the method of least squares (Collatz, 1960; Botha and Pinder, 1983; Allen *et al.*, 1988). The general approach may be outlined as follows: Consider the ordinary differential equation

$$G[y] - f(x) = 0 \quad \text{for} \quad x_0 \leq x \leq x_1 \qquad (2.3.1)$$

with boundary condition

$$B[y]=0 \quad \text{for} \quad x=x_0 \quad \text{and} \quad x=x_1 \tag{2.3.1a}$$

where $f(x)$ is a known function of x, and G and B denote differential operators of x. In general, G and B may be nonlinear operators, for example,

$$G=C_1\frac{d^2}{dx^2}+C_2\left(\frac{d}{dx}\right)^2+C_3$$

In linear boundary-value problems, G and B are linear differential operators, for example,

$$G=C_1\frac{d^2}{dx^2}+C_2\frac{d}{dx}+C_3$$

The C's may be functions of x. The general solution of eqn (2.3.1) is a function $y=y(x)$ that satisfies the differential equation.

In the method of weighted residuals, we may assume an approximate solution $\bar{y}$ of eqn (2.3.1) of the form

$$y\approx\bar{y}(x: a_1, a_2, a_3, \ldots, a_n)=\sum_{i=1}^{n} a_i\phi_i(x) \tag{2.3.2}$$

where the a_i are undetermined parameters, and the $\phi_i(x)$ are *trial functions* chosen so that $\bar{y}$ satisfies boundary conditions (the interior method). For example, if the boundary conditions are $y=0$ for $x=a$ and for $x=b$, a possible choice of $\phi_i(x)$ is

$$\phi_i(x)=(x-a)(x-b)x^{i-1} \tag{2.3.2a}$$

In some problems a boundary condition may be nonlinear (in y). Then it may not be possible to select $\phi_i(x)$ such that $\bar{y}$ satisfies this boundary condition, and we are led to the mixed method of determining the a_i. The residual $r(x)=B(\bar{y})$ formed by substituting $\bar{y}$ and its derivatives for y and its derivatives into the boundary condition (see eqn (2.3.1a)) is then not identically zero. We may obtain an equation in the a_i by liquidating the residual $r(x)$ in some way, for example, by arbitrarily setting it equal to zero. This equation must then be solved with the other equations by the method of weighted residuals.

For the *interior method*, the method of weighted residuals (Collatz (1960) calls it the orthogonality method) requires that the integral of the residual, $R(x)=G[\bar{y}]-f(x)$, appropriately weighted by