

# **ELEMENTS OF MICROWAVE ENGINEERING**

**R. CHATTERJEE, Ph.D.**

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# Preface

This text treats a subject that stems from Maxwell's mathematical formulation of the basic laws of electricity and magnetism, which led to the theoretical prediction of the reality of electromagnetic waves and which is of both theoretical elegance and growing practical utility. It is the outgrowth of a series of lectures on electromagnetic theory and applications and microwave engineering delivered over the last several years at the Indian Institute of Science, Bangalore. Most topics included here have also been successfully used in intensive courses specially organized for the professional engineers engaged in design, and research and development.

The volume is intended for the undergraduate in electronics, and electrical communication and electrical engineering as a comprehensive, up-to-date text, covering the fundamental principles of the various aspects of microwave engineering. It would be equally useful for the graduate in physics specializing in electronics.

Throughout, the theoretical principles and the analytical results have been presented in a form an average student can grasp and put into practical applications. The treatment is simple yet rigorous enough, and includes many worked examples illustrating the various analytical methods. No prior knowledge of mathematics, beyond that acquired by the engineering undergraduate or the graduate in physics, is required, and the chapters have been arranged on this basis. Owing to the limitations of space, the theoretical discussion and the more detailed parts of the applications have been curtailed to a certain extent in some places, but nothing of fundamental importance to a microwave engineer has been omitted. The reader wishing to supplement his knowledge of the subject is advised to consult the References/Suggested Reading provided.

The explosive growth of microwave engineering in the last few years makes it almost impossible to give a sufficiently thorough survey of all problems. An attempt has therefore been made to stress the basic scientific principles and to present in one volume several topics so that the reader can develop a good overall view of the subject. (A treatment of the advanced topics is given in the author's "Microwave Engineering: Special Topics" to be shortly published.) It is hoped that the book would benefit all those for whom it is intended.

I wish to thank Professor S. Dhawan, former Director, Professor S. Ramakrishna, former Chairman, Division of Electrical Sciences, and Professor S. Nagaraja, former Chairman, Department of Electrical Communication Engineering (all of the Indian Institute of Science) for their constant help and encouragement. I am indebted to the University Grants Commission for sponsoring a project for writing this text. I should also like to express my gratitude to Mr. N. Govindaraju and Mr. Varadaraja Iyengar for typing the manuscript, to Mr. R. Vijayendra for drawing the diagrams, and to Miss A. V. Ashajayanti, (late) Miss R. Sudha, and Dr. (Miss) Parveen Fatima Wahid for their assistance in preparing the manuscript.

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R. CHATTERJEE

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# 1 Introduction

## 1.1 Microwave Frequency Range

The discovery of Maxwell that light, by its very nature, is electromagnetic, was the starting point for the evolution of the concept of an electromagnetic spectrum that extends from d-c to  $\gamma$ -rays. The term *microwave frequencies* is very commonly used for those wavelengths measured from 30 cm–0.3 mm which correspond to the frequency range  $10^9$ – $10^{12}$  Hz. Since a large number of electronic communication systems utilize the space propagation path, and since a certain bandwidth is required for each transmission, the frequency spectrum of interest to communication engineers has become an international resource. According to the International Radio Consultative Committee (CCIR), the frequency ranges are as designated in Table 1.1. For convenience, the frequencies are also often designated in terms of bands (see Table 1.2).

## 1.2 Historical Resumé of Early Work on Microwaves

Hertz (1893) conducted a series of experiments at  $\lambda = 66$  cm, with a transmitter consisting of a parabolic mirror antenna which was fed by a dipole excited by spark discharges produced by an

Table 1.1 Designation of Frequency Ranges

Frequency ( $f$ )	Wavelength ( $\lambda$ )	Designation
<30 kHz	>10 km	VLF
30–300 kHz	10–1 km	LF
0.3–3 MHz	1–0.1 km	MF
3–30 MHz	100–10 m	HF
30–300 MHz	10–1 m	VHF
300–3000 MHz	1–0.1 m	UHF
3–30 GHz	100–10 mm	SHF
30–300 GHz	10–1 mm	EHF

induction coil, and with a receiver comprising a similar antenna with a dipole whose output was passed on to a spark-gap detector placed behind the mirror. These early experiments established beyond doubt *action at a distance* and proved that this action was communicated to a distance

Table 1.2 Bands for Frequency Ranges

<i>Band designation</i>	<i>Frequency range (MHz)</i>
P	225-390
L	390-1550
S	1550-5200
X	5200-10,900 <sup>a</sup>
K	10,900-36,000
Q	36,000-46,000
V	45,000-56,000

by wave motion. This justified Maxwell's theoretical prediction that the waves responsible for optical phenomena are electromagnetic. Hertz's work on reflection, diffraction, polarization, and measurement of wavelength by interference technique may be said to have led to the discovery of radio frequency optics, where the phenomenology of optics can be represented by microwaves. Righi (1897) performed many quasi-optical experiments at X- and S-bands and thus firmly laid the foundation of microwave optics. Lodge and Howard (1889) constructed a cylindrical lens of pitch. Lodge (1897, and 1898 and 1899) was also successful in establishing the mode property of propagation in a hollow tube and transmission of signals through space without wires. Bose (1895, 1897, and 1898a-1898c) conducted several microwave experiments at 5 mm with apparatus of his own design such as microwave spectrometers, diffraction gratings, polarimeters, spark generators, and coherer detectors. For a description of these experiments, see Ramsay (1958). Thus, it is evident that the pioneering work before 1900 by Hertz, Lodge, and Bose laid the foundations of modern microwave engineering.

### 1.3 Correspondence between Field and Circuit Concepts

Since the wavelengths at microwave frequencies are of the same order of magnitude as the dimensions of circuit devices, and the time of propagation of electrical effects from one part of the circuit to the other is comparable to the period of oscillating currents and charges, conventional circuit concepts of currents and voltages need to be replaced by field concepts. At microwave frequencies, the difficulty in applying circuit concepts is obvious when the potential difference between two points ordinarily means the line integral of the electric field strength, namely,

$$\int E \cdot ds,$$

taken at one instant of time, along some paths joining the two points. This concept is unique and useful only if the value of this line integral is independent of the path. But if the path length is not small compared to the wavelength, the line integral is not, in general, independent of the path, and hence the significance of the term *voltage* is lost. This suggests that, at microwave frequencies, we have to deal with electric and magnetic fields instead of with voltage and current.

Maxwell's field equations are generalizations of Faraday's laws of induction, Ampere's circuit law, and Gauss' law. These equations established that *magnetic flux source* does not exist. Hence, a close correspondence between circuit concepts and field concepts can be established: For example, the field equation  $\nabla \times E = -\partial B/\partial t$  corresponds to the circuit equation  $\mathcal{E} = -\partial \psi/\partial t$ ,  $\psi$  being the magnetic flux; the power flow given by the equation  $P = E \times H$  corresponds to the circuit concept of power, namely,  $P = VI$ .

Maxwell's field equations, their solutions, and their applications to several practical and useful problems form the subject matter of this text. The field and circuit concepts are used to study the characteristics of transmission lines, waveguides, and passive microwave components.

## 1.4 Some Useful Applications of Microwaves

Since the transit-time effects of electrons were the major limitations of the conventional high-frequency tubes, these conventional tubes could not succeed in the microwave region. These handicaps were overcome by introducing the concept of interaction of electron beams with electromagnetic fields, resulting in the development of magnetrons, klystrons, and travelling-wave tubes, which made the evolution of radar possible. These developments during World War II opened up new vistas for the extensive application of microwaves, not only to the technological fields such as defence, but to areas of civilian interest, e.g., microwave communication relay links, satellite communication, and domestic appliances, for instance, microwave ovens. Microwaves also find extensive application in pure scientific fields such as radio astronomy, spectroscopy, and materials research which led to the development of solid-state microwave generators, e.g., masers, coherent light generators such as lasers, and ferrite microwave devices.

The principles of microwave tubes and their modern solid-state counterparts are discussed in Chapter 10 in a language that can be understood not only by undergraduates in electronics and electrical communication engineering, but also by post-graduate students in physics.

Microwaves are currently used in India in the following areas: defence, post and telegraph, railways, civil aviation, space communication, police, and radio astronomy. It may therefore be stated that a comprehensive course in microwave engineering for the undergraduate and post-graduate level students should include the following topics:

- (i) Maxwell's field equations and their solutions
- (ii) Transmission lines and waveguides
- (iii) Microwave networks
- (iv) Microwave generators, including solid-state devices
- (v) Microwave antennas
- (vi) Microwave measurements
- (vii) Other related topics.

All these topics are discussed in this text. The treatment of some of the topics may be found to be somewhat condensed due to limitations of space. However, we have endeavoured to emphasize the fundamental principles rather than details, with the conviction that, if the fundamental principles are properly grasped, the details can be learned by consulting the proper references.

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# 2 Mathematical Review

## 2.1 Vector Analysis

*Vector* is a name given to physical quantities such as force, velocity, and field intensity which can be defined uniquely only when their magnitudes and directions are specified. A vector is represented graphically by a directed segment  $\vec{PQ}$  or  $A$  (see Fig. 2.1) whose length is proportional to

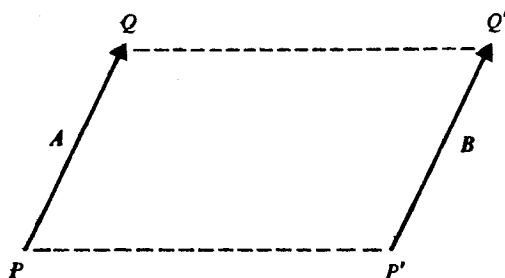


Fig. 2.1 Equal vectors.

the magnitude of the vector and whose direction is the same as that of  $\vec{PQ}$ . Two parallel vectors  $\vec{PQ}$  ( $A$ ) and  $\vec{P'Q'}$  ( $B$ ) are considered equal if they have the same magnitude and direction.

A *scalar* is a quantity which can be specified only by its magnitude and does not have a direction. Examples of scalar are temperature, speed, and mass.

The common laws of addition, subtraction, multiplication, and division which are applicable to scalars are not applicable to vectors. Throughout this text, therefore, we shall denote a vector by  $F$  and a scalar by  $F$  and the magnitude of  $F$  by  $|F|$  or  $F$ .

### Addition and Subtraction of Vectors

The sum of the vectors  $A$  and  $B$  is given by the diagonal of the parallelogram constructed with these vectors as the adjacent sides (see Fig. 2.2). Since  $\vec{PQ} + \vec{QP} = 0$ ,

$$\vec{QP} = -\vec{PQ}.$$

Hence,  $A - B = \vec{PQ} - \vec{PR} = \vec{PQ} + \vec{RP} = \vec{PQ} + \vec{PR}' = \vec{PS}' = \vec{RQ}$ , as shown in Fig. 2.3.

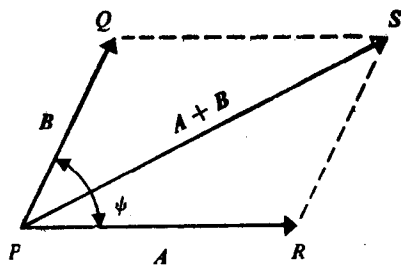


Fig. 2.2 Addition of two vectors.

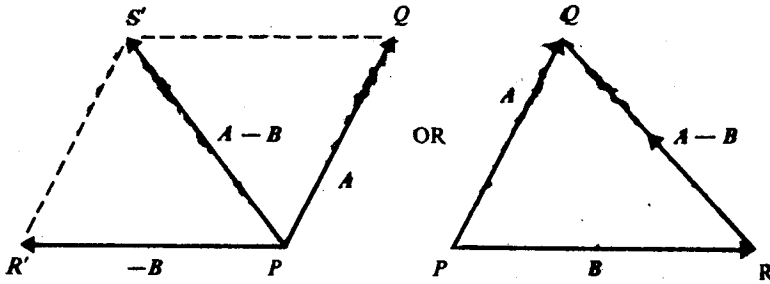


Fig. 2.3 Subtraction of two vectors.

### Scalar and Vector Products

The *scalar product* of the vectors  $A$  and  $B$  is defined as the product of their magnitudes  $A$  and  $B$  and the cosine of the angle  $\psi$  between them. Thus

$$A \cdot B = AB \cos \psi. \quad (2.1)$$

If the scalar product of two vectors is zero, then they are perpendicular to each other.

Scalar multiplication obeys the commutative and distributive laws

$$A \cdot B = B \cdot A, \quad (A + B) \cdot C = A \cdot C + B \cdot C. \quad (2.2)$$

The component of a given vector  $A$  in a particular direction defined by a unit vector  $u$  in that direction is the scalar product of  $A$  and  $u$ , that is,  $A \cdot u$ ; or, in other words, the component is the projection of the vector  $A$  on  $u$ . The cartesian components of a vector  $A$ , drawn from a point  $P(x_1, y_1, z_1)$  to a point  $Q(x_2, y_2, z_2)$ , along the positive directions of the  $x$ -,  $y$ -, and  $z$ -axis are given by  $x_2 - x_1$ ,  $y_2 - y_1$ , and  $z_2 - z_1$ . If  $l$  is the length of the vector and  $\alpha, \beta, \gamma$  are the angles the vector makes with the coordinate axes, then

$$PQ_x = A_x = x_2 - x_1 = l \cos \alpha, \quad (2.3)$$

$$PQ_y = A_y = y_2 - y_1 = l \cos \beta, \quad (2.4)$$

$$PQ_z = A_z = z_2 - z_1 = l \cos \gamma, \quad (2.5)$$

$$|A| = (A_x^2 + A_y^2 + A_z^2)^{1/2}. \quad (2.6)$$

The scalar product  $A \cdot B$  of the two vectors  $A$  and  $B$  can be expressed as

$$A \cdot B = A_x B_x + A_y B_y + A_z B_z, \quad (2.7)$$

and the cosine of the angle  $\psi$  between the two vectors is given by

$$\cos \psi = \cos \alpha_A \cos \alpha_B + \cos \beta_A \cos \beta_B + \cos \gamma_A \cos \gamma_B, \quad (2.8)$$

where  $(\alpha_A, \beta_A, \gamma_A)$  and  $(\alpha_B, \beta_B, \gamma_B)$  are respectively the angles  $A$  and  $B$  make with the three co-ordinate axes  $x, y$ , and  $z$ .

The *vector product*  $A \times B$  of the vectors  $A$  and  $B$  is defined as a vector perpendicular to both  $A$  and  $B$ , pointing in the direction towards which a right-handed screw would advance if turned



from  $A$  to  $B$  through the smaller angle (see Fig. 2.4). The magnitude of the vector product is the

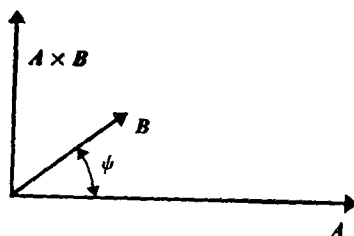


Fig. 2.4 Vector product.

product of the magnitudes of  $A$  and  $B$  and of the sine of the angle between them, that is, the area of the parallelogram constructed with  $A$  and  $B$  as the adjacent sides.

For vector products, we have

$$A \times B = -B \times A, \quad (2.9)$$

$$(A + B) \times C = A \times C + B \times C. \quad (2.10)$$

The components of the vector product  $A \times B$  in the cartesian coordinates  $x, y, z$  are expressed as

$$(A \times B)_x = A_y B_z - A_z B_y, \quad (2.11)$$

$$(A \times B)_y = A_z B_x - A_x B_z, \quad (2.12)$$

$$(A \times B)_z = A_x B_y - A_y B_x. \quad (2.13)$$

If  $A$  and  $B$  are expressed as

$$A = u_x A_x + u_y A_y + u_z A_z, \quad (2.14)$$

$$B = u_x B_x + u_y B_y + u_z B_z, \quad (2.15)$$

then

$$A \times B = \begin{vmatrix} u_x & u_y & u_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \quad (2.16)$$

### Functions of Position

A *function of position* or *point function* is a function  $f(x, y, z)$  depending only on the position of points. The *loci* of equal values of a point function are called *level surfaces* or *contour surfaces*. Some level surfaces have special names, e.g., equipotential, isothermal, and isobar surfaces. Figure 2.5 illustrates how two-dimensional point functions may be represented graphically by drawing contour lines.

In Fig. 2.5a, the solid curves are the contour lines  $x^2 + y^2 = \text{constant } r^2$ , and the dashed lines are the contour curves  $y/x = \tan^{-1} \phi$  constant. In Fig. 2.5b, the solid curves are the contour lines  $\log(\rho_1/\rho_2) = u$ , where  $\rho_1$  and  $\rho_2$  are the distances from two fixed points  $A$  and  $B$ , and the dashed curves are the contour curves for  $\theta = \text{constant}$ ,  $\theta$  being the angle made by  $BP$  with  $PA$ .