Handbook of APPLIED MATHEMATICS

Selected Results and Methods

Second Edition

edited by
Carl E. Pearson

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Preface

Most of the topics in applied mathematics dealt with in this handbook can be grouped rather loosely under the term *analysis*. They involve results and techniques which experience has shown to be of utility in a very broad variety of applications.

Although care has been taken to collect certain basic results in convenient form, it is not the purpose of this handbook to duplicate the excellent collections of tables and formulas available in the *National Bureau of Standards Handbook of Mathematical Functions* (AMS Series 55, U.S. Government Printing Office) and in the references given therein. Rather, the emphasis in the present handbook is on technique, and we are indeed fortunate that a number of eminent applied mathematicians have been willing to share with us their interpretations and experiences.

To avoid the necessity of frequent and disruptive cross-referencing, it is expected that the reader will make full use of the index. Moreover, each chapter has been made as self-sufficient as is feasible. This procedure has resulted in occasional duplication, but as compensation for this the reader may appreciate the availability of different points of view concerning certain topics of current interest.

As editor, I would like to express my appreciation to the contributing authors; to the reviewers, to the editorial staff of the publisher, and to the many secretaries and typists who have worked on the manuscript: without the partnership of all of these people, this handbook would not have been possible.

CARL E. PEARSON

Changes in the Second Edition:

Some material less directly concerned with technique has been omitted or consolidated. Two new chapters, on Integral Equations, and Mathematical Modelling, have been added. Several other chapters have been revised or extended, and known misprints have been corrected.

Contents

1

Formulas from Algebra, Trigonometry and Analytic Geometry

H. Lennart Pearson*

1.1 THE REAL NUMBER SYSTEM

Readers wishing a logical development of the real number system are directed to the references at the end of the chapter. Here the real numbers are considered to be the set of all terminating and nonterminating decimals with addition, subtraction, multiplication and division (except by zero) defined as usual. Addition and multiplication satisfy

the Commutative Law

$$a + b = b + a \qquad (1.1\text{-}1)$$

$$ab = ba \qquad (1.1\text{-}2)$$

the Associative Law

$$a + (b + c) = (a + b) + c \qquad (1.1\text{-}3)$$

$$a(bc) = (ab)c \qquad (1.1\text{-}4)$$

the Distributive Law

$$a(b + c) = ab + ac \qquad (1.1\text{-}5)$$

The real numbers are an ordered set, i.e., given any two real numbers a and b, one

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of the following must hold:

$$a < b \qquad a = b \qquad a > b \tag{1.1-6}$$

The real numbers fall into two classes, the rational numbers and the irrational numbers. A number is rational if it can be expressed as the quotient of two integers. Division of one integer by another to give a decimal shows that a rational number is either a terminating or repeating decimal. Conversely, any repeating decimal is a rational number, as indicated by the following example:

$$\begin{aligned} &4.328328328 \cdots \\ &= 4 + \frac{328}{10^3} + \frac{328}{10^6} + \frac{328}{10^9} + \cdots \\ &= \frac{4324}{999} \quad \text{by summing the geometric series} \end{aligned}$$

Also note for example $8 = 7.999999 \cdots$.

Non-repeating decimals correspond to irrational numbers.

A set of numbers is said to be bounded above if there exists a number M such that every member of the set is $\leq M$. The smallest such M is called the least upper bound of the given set. Similarly a set of numbers is bounded below if there exists a number Q such that every member of the set is $\geq Q$, and the greatest lower bound is the largest such Q. Any non-empty set of numbers which is bounded above has a least upper bound, and similarly, if it is bounded below it has a greatest lower bound.

1.2 THE COMPLEX NUMBER SYSTEM

1.2.1 Definition, Real and Imaginary Parts

Two definitions will be given. First, any number of the form $a + ib$ where a and b are any real numbers and $i^2 = -1$ is called a complex number. The number a is called the real part of the complex number and b is called the imaginary part. If $a + ib$ is denoted by the single letter z, then the notation $a = R(z)$, $b = I(z)$ is used.

A second definition, more modern in character, is to define the complex numbers as the set of all ordered pairs (a, b) of real numbers satisfying

$$(a, b) + (c, d) = (a + c, b + d) \tag{1.2-1}$$

$$(a, b)(c, d) = (ac - bd, ad + bc) \tag{1.2-2}$$

In particular,

$$(a, 0) + (c, 0) = (a + c, 0)$$

$$(a, 0)(c, 0) = (ac, 0)$$

so $(a, 0)$ may be identified with the real number a. By (1.2-2), $(0, 1)(0, 1) = (-1, 0)$ and if i is used for the complex number $(0, 1)$ then

$$a + ib = (a, 0) + (0, 1)(b, 0) = (a, 0) + (0, b) = (a, b)$$

and the two definitions are seen to be equivalent.

1.2.2 Conjugate, Division, Modulus and Argument

The conjugate $\bar{z}$ of the complex number $z = a + ib$ is $\bar{z} = a - ib$.

$$z + \bar{z} = 2R(z) \qquad z - \bar{z} = 2iI(z) \tag{1.2-3}$$

Also,

$$\left.\begin{aligned} \overline{z_1 + z_2} &= \bar{z}_1 + \bar{z}_2 \\ \overline{z_1 z_2} &= (\bar{z}_1)(\bar{z}_2) \\ \overline{\left(\frac{z_1}{z_2}\right)} &= \frac{\bar{z}_1}{\bar{z}_2} \end{aligned}\right\} \tag{1.2-4}$$

Division of one complex number by another is illustrated by example:

$$\frac{3 + 2i}{4 - 3i} = \frac{(3 + 2i)(4 + 3i)}{(4 - 3i)(4 + 3i)} = \frac{6 + 17i}{25} = \frac{6}{25} + i\,\frac{17}{25}$$

The **modulus**, or **absolute** value, of the complex number $z = a + ib$ is $|z| = \sqrt{a^2 + b^2}$ and the argument, or amplitude, of z is arc tan (b/a). Note that

$$z\bar{z} = |z|^2, \quad |z_1 z_2| = |z_1||z_2|, \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \tag{1.2-5}$$

1.2.3 The Argand Diagram

A one-to-one correspondence exists between the complex numbers and the points in a plane:

$a + ib \longleftrightarrow$ point in 2-space with coordinates (a, b)

1.2.4 Polar Form, de Moivre's Formula

Introducing polar coordinates in the plane, $x = r\cos\theta$, $y = r\sin\theta$ the complex number $x + iy$ can be written in the polar form:

$$x + iy = r(\cos\theta + i\sin\theta) \tag{1.2-6}$$

where $r = \sqrt{x^2 + y^2}$ is the modulus of z and $\theta = \text{arc tan}\,(y/x)$ is the argument of z.

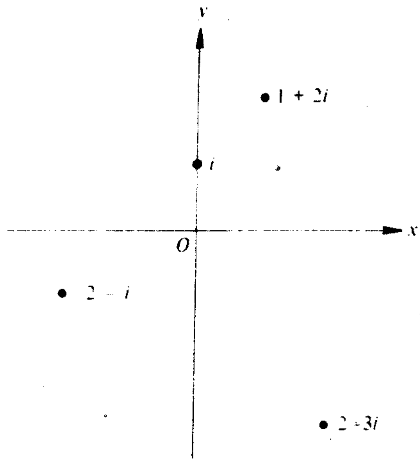

Fig. 1.2-1 Argand diagram: representation of complex numbers in the plane.

Multiplication and division of numbers in polar form yields

$$[r_1(\cos\theta_1 + i\sin\theta_1)]\,[r_2(\cos\theta_2 + i\sin\theta_2)] = r_1 r_2\,[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \qquad (1.2\text{-}7)$$

$$\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}\,[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \qquad (1.2\text{-}8)$$

and extending the multiplication to n equal factors gives de Moivre's Formula:

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta) \qquad (1.2\text{-}9)$$

1.2.5 The n^{th} Roots of a Complex Number

Given $z = r(\cos\theta + i\sin\theta)$,

$$z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right] \quad k = 0, 1, 2, 3, \ldots, n-1 \qquad (1.2\text{-}10)$$

which leads to a more general form of de Moivre's Formula

$$z^{m/n} = r^{m/n}\left[\cos\frac{m}{n}(\theta + 2k\pi) + i\sin\frac{m}{n}(\theta + 2k\pi)\right]$$

$$k = 0, 1, 2, \ldots, n-1;\ m \text{ and } n \text{ having no factors in common} \qquad (1.2\text{-}11)$$

1.3 INEQUALITIES

1.3.1 Rules of Operation, the Triangle Inequality

For a, b and c real numbers,

if $a < b$ and $b < c$ then $a < c$

if $a < b$ then $a + c < b + c$

if $a < b$ then $ac < bc$ if $c > 0$

if $a < b$ then $ac > bc$ if $c < 0$.

Also $|a + b| \leqslant |a| + |b|$ which is the Triangle Inequality, where $|a|$ is a if $a \geqslant 0$ and is $-a$ if $a \leqslant 0$.

1.3.2 The Inequalities of Hölder, Cauchy-Schwarz and Minkowski

Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be any real or complex numbers.

Hölder's Inequality

$$\left|\sum_{k=1}^{n} a_k b_k\right| \leqslant \sum_{k=1}^{n} |a_k b_k| \leqslant \left(\sum_{k=1}^{n} |a_k|^{\lambda}\right)^{1/\lambda} \left(\sum_{k=1}^{n} |b_k|^{\alpha}\right)^{1/\alpha}$$

$$\text{where } \lambda > 1 \text{ and } \alpha = \frac{\lambda}{\lambda - 1} \qquad (1.3\text{-}1)$$

Cauchy-Schwarz Inequality

$$\left|\sum_{k=1}^{n} a_k b_k\right|^2 \leqslant \left(\sum_{k=1}^{n} |a_k|^2\right) \left(\sum_{k=1}^{n} |b_k|^2\right) \qquad (1.3\text{-}2)$$

Minkowski Inequality

$$\left(\sum_{k=1}^{n} |a_k + b_k|^{\lambda}\right)^{1/\lambda} \leqslant \left(\sum_{k=1}^{n} |a_k|^{\lambda}\right)^{1/\lambda} + \left(\sum_{k=1}^{n} |b_k|^{\lambda}\right)^{1/\lambda} \quad \text{where } \lambda \geqslant 1 \qquad (1.3\text{-}3)$$

1.4 POWERS AND LOGARITHMS

1.4.1 Rules of Exponents

Let a and b be any positive real numbers and let m and n be positive integers. Then a^n is defined to be the result obtained by multiplying a by itself n times. Then

$$a^n a^m = a^{n+m} \qquad (1.4\text{-}1)$$

$$(a^n)^m = a^{nm} \qquad (1.4\text{-}2)$$

$$(ab)^n = a^n b^n \qquad (1.4\text{-}3)$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \qquad (1.4\text{-}4)$$

A meaning for a^0 consistent with these rules is obtained by considering $a^n a^0 = a^{n+0} = a^n$, so that $a^0 = 1$; then $a^{-n}a^n = a^{-n+n} = a^0 = 1$ so $a^{-n} = 1/a^n$.

$$\frac{a^n}{a^m} = a^{n-m} \tag{1.4-5}$$

1.4.2 Radicals, Fractional Exponents

Any number a such that $a^n = b$, where n is a positive integer, is called an n^{th} root of b. Any number b has exactly n, n^{th} roots (1.2.5). The principal root of a positive number is defined to be the positive root and the principal n^{th} root (n odd) of a negative number is the negative root. Principal root is not defined when b is negative and n is even. The radical symbol $\sqrt[n]{\ }$ is defined to be the principal n^{th} root, whenever that is defined, and to stand for any one of the n roots if there is no principal root.

Example:

$$\sqrt{4} = 2, \quad \sqrt[5]{-32} = -2, \quad \sqrt{-9} = \pm 3i$$

$a^{1/n}$ is defined to be the same as $\sqrt[n]{a}$

Example:

$$4^{1/2} = 2, \quad (-32)^{1/5} = -2, \quad (-9)^{1/2} = \pm 3i$$

Also

$$a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m} \qquad m, n \text{ positive integers} \tag{1.4-6}$$

Finally the definition of a^α, α irrational, will be illustrated by example. Consider 2^π, where $\pi = 3.141592654\ldots$. Then 2^π is defined as the limit of the sequence $2^3, 2^{3.1}, 2^{3.14}, 2^{3.141}, 2^{3.1415}, 2^{3.14159}, 2^{3.141592}, \ldots$, each term of which is defined by the above.

1.4.3 Definitions and Rules of Operation for Logarithms

For any positive number n and any positive number a except 1, there exists a unique real number x such that $n = a^x$. x is called the logarithm of n to the base a. This is written either as above or as $x = \log_a n$. Logarithms have the following properties and rules of operation:

$$\log_a 1 = 0 \qquad \log_a a = 1 \qquad a^{\log_a n} = n \tag{1.4-7}$$

$$\log_a (m \cdot n) = \log_a m + \log_a n \tag{1.4-8}$$