TREATISE ON ANALYSIS

J. DIEUDONNÉ

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ANALYSIS

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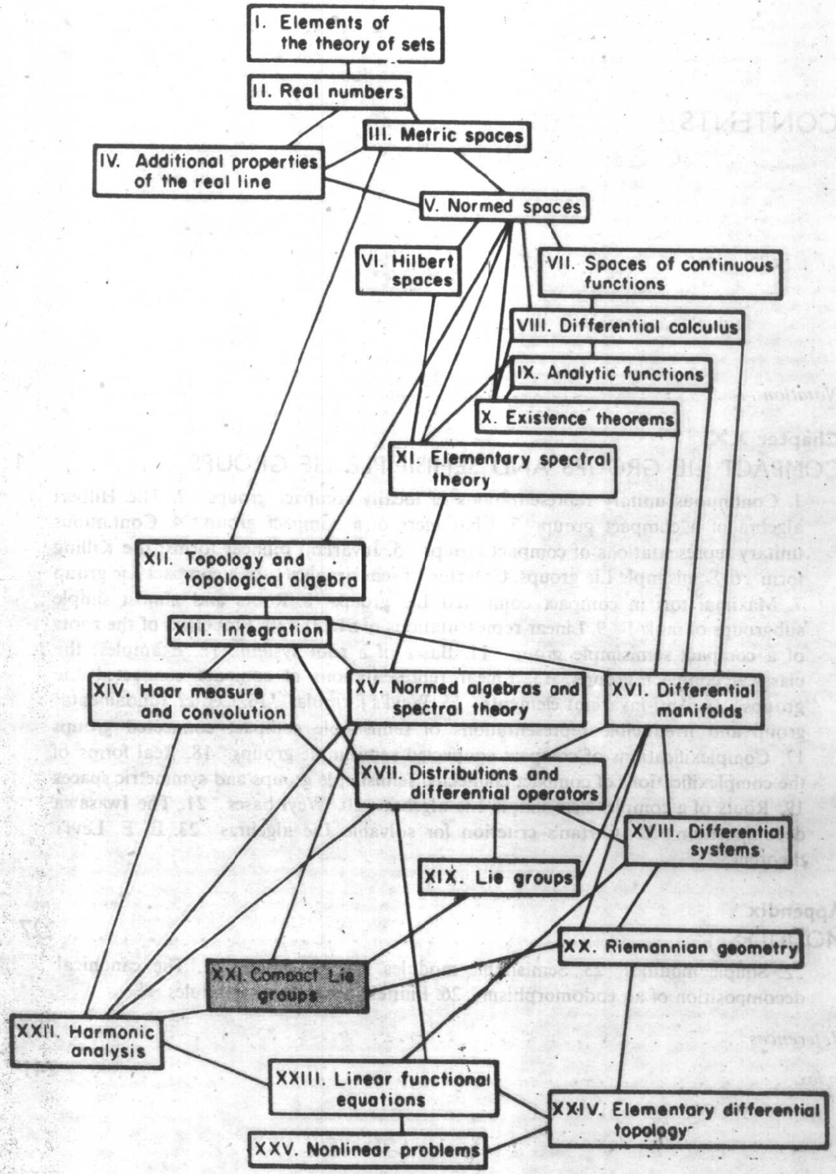

NOTATION

In the following definitions, the first number indicates the chapter in which the notation is introduced, and the second number indicates the section within the chapter.

$U(\mu)$	$\int U(s)\, d\mu(s)$, where U is a continuous unitary representation of a group G and μ is a bounded measure on G: 21.1
$U(f)$, $U(\tilde{f})$	$\int f(s)\, U(s)\, d\beta(s)$, where β is a Haar measure on G, and $f \in \mathscr{L}^1_{\mathbf{C}}(G, \beta)$: 21.1
U_{ext}	mapping $\mu \mapsto U(\mu)$: 21.1
$R(s)$	left regular representation $\tilde{f} \mapsto (\varepsilon_s * f)\tilde{}$: 21.1
$U_1 \oplus U_2$	direct sum of two continuous linear representations: 21.1
$U^{(\mathbf{R})}$, $U^{(\mathbf{H})}$	real (resp. quaternionic) linear representation corresponding to a complex linear representation U: 21.1, Problem 9
$\mathfrak{a}_\rho$ ($\rho \in \mathbf{R}(G)$)	minimal two-sided ideals of the complete Hilbert algebra $L^2_{\mathbf{C}}(G)$, for G compact: 21.2
u_ρ	identity element of $\mathfrak{a}_\rho$: 21.2
n_ρ	the integer such that $\mathfrak{a}_\rho$ is isomorphic to $\mathbf{M}_{n_\rho}(\mathbf{C})$: 21.2
$m^{(\rho)}_{jk}$	elements of $\mathfrak{a}_\rho$: 21.2
$M_\rho(s)$	the matrix $(n_\rho^{-1} m^{(\rho)}_{ij}(s))$: 21.2
ρ_0	index of the trivial ideal $\mathfrak{a}_{\rho_0} = \mathbf{C}$: 21.2

χ_ρ	$n_\rho^{-1}u_\rho$: 21.3
$\bar{\rho}$	the index such that $\chi_{\bar{\rho}} = \overline{\chi_\rho}$: 21.3
$\mathrm{cl}(V)$	class $\sum_{\rho \in R} d_\rho \cdot \rho$ of a finite-dimensional linear representation: 21.4
$\mathbf{Z}^{(R)}$, $\mathbf{Z}^{(R(G))}$	ring of classes of continuous linear repsentations of G: 21.4
B_U	bilinear form $(\mathbf{u}, \mathbf{v}) \mapsto \mathrm{Tr}(U_*(\mathbf{u}) \circ U_*(\mathbf{v}))$ associated with a linear representation U of a Lie group: 21.5
B_ρ	bilinear form $(\mathbf{u}, \mathbf{v}) \mapsto \mathrm{Tr}(\rho(\mathbf{u}) \circ \rho(\mathbf{v}))$ associated with a homomorphism of Lie algebras $\rho : \mathfrak{g} \to \mathfrak{gl}(F)$: 21.5
$B_\mathfrak{a}$	Killing form $(\mathbf{u}, \mathbf{v}) \mapsto \mathrm{Tr}(\mathrm{ad}(\mathbf{u}) \circ \mathrm{ad}(\mathbf{v}))$ of a Lie algebra $\mathfrak{a}$: 21.5
Γ_T	kernel $\exp_T^{-1}(e)$ of the exponential $\exp_T$: $\mathfrak{t} \to T$, where $\mathfrak{t}$ is the Lie algebra of the torus T: 21.7
Γ_T^*	dual of the lattice Γ_T, in $\mathfrak{t}^*$: 21.7
W(G, T), W(G), W	Weyl group $\mathcal{N}(T)/T$, where T is a maximal torus of G: 21.7
$w.\lambda$	${}^t w^{-1}(\lambda)$, for $w \in W$ and $\lambda \in \mathfrak{t}^*$: 21.8
$\mathbf{S}$(G, T), $\mathbf{S}$(G), $\mathbf{S}$	set of roots of G with respect to T: 21.8
$\mathfrak{g}_\alpha$	subspace of $\mathfrak{g}_{(C)}$ consisting of the vectors $\mathbf{x}$ such that $[\mathbf{u}, \mathbf{x}] = \alpha(\mathbf{u})\mathbf{x}$ for all $\mathbf{u} \in \mathfrak{t}$: 21.8
U_α	subgroup $\chi_\alpha^{-1}(1)$ of T, where $\chi_\alpha(\exp(\mathbf{u})) = e^{\alpha(\mathbf{u})}$ for $\mathbf{u} \in \mathfrak{t}$: 21.8
$\mathfrak{u}_\alpha$	hyperplane $\alpha^{-1}(0)$ in $\mathfrak{t}$: 21.8
s_α	element of W acting on $\mathfrak{t}$ by reflection in the hyperplane $\mathfrak{u}_\alpha$: 21.8
L_m	simple $U(\mathfrak{sl}(2, C))$-module of dimension $m + 1$: 21.9
$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathbf{S}} \mathfrak{g}_\alpha$	root decomposition of a complex semisimple Lie algebra $\mathfrak{g}$: 21.10 and 21.20
$\mathbf{h}_\alpha^0$	element of $\mathfrak{h}$ such that $\alpha(\mathbf{h}) = \Phi(\mathbf{h}, \mathbf{h}_\alpha^0)$: 21.10
$\mathbf{h}_\alpha$	element of $\mathfrak{h}$ such that $\alpha(\mathbf{h}_\alpha) = 2$ and $\mathbf{h}_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$: 21.10
$\mathbf{x}_\alpha$, $\mathbf{x}_{-\alpha}$	elements of $\mathfrak{g}_\alpha$, $\mathfrak{g}_{-\alpha}$, respectively, such that $[\mathbf{x}_\alpha, \mathbf{x}_{-\alpha}] = \mathbf{h}_\alpha$: 21.10

σ_α	bijection $\lambda \mapsto \lambda - \lambda(\mathbf{h}_\alpha)\alpha$ of $\mathfrak{h}^*$ onto itself: 21.10
$\mathfrak{s}_\alpha$	Lie subalgebra $\mathbf{C}\mathbf{h}_\alpha \oplus \mathbf{C}\mathbf{x}_\alpha \oplus \mathbf{C}\mathbf{x}_{-\alpha}$: 21.10
$N_{\alpha,\beta}$	numbers such that $[\mathbf{x}_\alpha, \mathbf{x}_\beta] = N_{\alpha,\beta}\mathbf{x}_{\alpha+\beta}$ when $\alpha + \beta \in \mathbf{S}$: 21.10
D(G)	union of the hyperplanes in $\mathfrak{t}$ with equations $\alpha(\mathbf{u}) = 2\pi i n$, $n \in \mathbf{Z}$: 21.10, Problem 2
σ_α	bijection $\lambda \mapsto \lambda - v_\alpha(\lambda)\alpha$, for a reduced root system $\mathbf{S}$ in F: 21.11
$W_{\mathbf{S}}$	Weyl group of $\mathbf{S}$, generated by the σ_α: 21.11
$n(\alpha, \beta)$	Cartan integers $v_\beta(\alpha) = 2(\beta\|\alpha)/(\beta\|\beta)$ for $\alpha, \beta \in \mathbf{S}$: 21.11
$\mathbf{S}_{\mathbf{x}}^+$	set of $\alpha \in \mathbf{S}$ such that $\alpha(\mathbf{x}) > 0$: 21.11
$\mathbf{B}_{\mathbf{x}}$	basis of $\mathbf{S}$, namely the set of indecomposable elements of $\mathbf{S}_{\mathbf{x}}^+$: 21.11
$\mathbf{S}^+$	set of positive roots, relative to a basis $\mathbf{B}$ of $\mathbf{S}$: 21.11
$\mathbf{S}^\vee$	root system formed by the $v_\alpha \in F^*$: 21.11
$\mathbf{B}^\vee$	basis of $\mathbf{S}^\vee$ consisting of the v_α, $\alpha \in \mathbf{B}$: 21.11
δ	$\frac{1}{2}\sum_{\lambda \in \mathbf{S}+} \lambda$: 21.11
ε_r	linear form on $\mathfrak{t} = \bigoplus_{s=1}^{n} \mathbf{R}iE_{ss} < \mathbf{M}_n(\mathbf{C})$ such that $\varepsilon_r(iE_{ss}) = i\delta_{rs}$: 21.12
$\mathbf{Sp}(2n, \mathbf{C})$, $\mathfrak{sp}(2n, \mathbf{C})$	complex symplectic group and its Lie algebra: 21.12
$\mathbf{SO}(m, \mathbf{C})$, $\mathfrak{so}(m, \mathbf{C})$	complex special orthogonal group and its Lie algebra: 21.12
A_n, B_n, C_n, D_n	Lie algebras of the classical groups: 21.12
P(G, T), P(G), P	lattice $2\pi i\Gamma_{\mathrm{T}}^*$ of weights of G with respect to T: 21.13
e^p, $s \mapsto e^{p(s)}$	character $\exp(\mathbf{u}) \mapsto e^{p(\mathbf{u})}$ of T, where $p \in$ P: 21.13
$S(\Pi)$	$\sum_{p \in \Pi} e^p$, where Π is an orbit of the Weyl group W in P: 21.13
$\mathbf{Z}[P]^W$	set of W-invariant elements of $\mathbf{Z}$[P]: 21.13

$\mathbf{h}_j$	$\mathbf{h}_{\beta_j}$, where $\{\beta_1, \ldots, \beta_l\}$ is a basis of S: 21.14
P(g)	set of $\lambda \in t^*_{(C)}$ such that $\lambda(\mathbf{h}_\alpha) \in \mathbf{Z}$ for all $\alpha \in \mathbf{S}$, or equivalently such that $\lambda(\mathbf{h}_j) \in \mathbf{Z}$ for $1 \leq j \leq l$: 21.14
C(g), C	Weyl chamber in it^*, consisting of the λ such that $\lambda(\mathbf{h}_j) > 0$ for $1 \leq j \leq l$: 21.14
$\lambda \leq \mu$	order relation on it^*, equivalent to $\lambda = \mu$ or $\mu - \lambda = \gamma + \sum_{j=1}^{l} c_j \beta_j$, with $\gamma \in ic^*$ and $c_j \geq 0$ and not all zero: 21.14
s_j	reflection s_{β_j}: $\lambda \mapsto \lambda - \lambda(\mathbf{h}_j)\beta_j$ for $1 \leq j \leq l$: 21.14
H_α	hyperplane in it^* with equation $\lambda(\mathbf{h}_\alpha) = 0$: 21.14
$\mathbf{Z}[P]^{aW}$	set of W-anti-invariant elements of $\mathbf{Z}$[P]: 21.14
$J(e^p)$	$\sum_{w \in W} \det(w) e^{w \cdot p}$, where $p \in P$: 21.14
P_{reg}	set of weights $\lambda \in P$ which are regular linear forms: 21.14
$S(p)$	$S(\Pi)$, where Π is the W-orbit of $p \in P \cap \bar{C}$: 21.14
Δ	$J(e^\delta) = \prod_{\alpha \in \mathbf{S}^+} (e^{\alpha/2} - e^{-\alpha/2})$: 21.14
T_{reg}	set of regular points of the maximal torus $T \subset G$: 21.15
$\upsilon_G, \upsilon_T, \upsilon_{G/T}$	invariant volume-forms on G, T and G/T: 21.15
$m_G, m_T, m_{G/T}$	invariant measures corresponding to the volume-forms $\upsilon_G, \upsilon_T, \upsilon_{G/T}$: 21.15
$\mu = n_1\beta_1 + \cdots + n_l\beta_l$	highest root in $\mathbf{S}$, relative to the basis $\mathbf{B} = \{\beta_1, \ldots, \beta_l\}$: 21.15, Problem 10
W_a	affine Weyl group: 21.15, Problem 11
$\mathfrak{u}_{\alpha,k}$	hyperplane with equation $\alpha(\mathbf{u}) = 2\pi k$ in it: 21.15, Problem 11
$\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_l\}$	basis of it dual to $\{\beta_1, \beta_2, \ldots, \beta_l\}$: 21.15, Problem 11
Q(g)	sublattice P(G/Z) of P(G) generated by the roots $\alpha \in \mathbf{S}$: 21.16
ϖ_j	fundamental weights ($1 \leq j \leq l$) relative to the basis $\mathbf{B}$ of $\mathbf{S}$: 21.16
Spin (m)	simply connected covering group of **SO**(m) ($m \geq 3$): 21.16

$\mathfrak{a}(E)$	set of self-adjoint automorphisms of E: 21.17
$\mathfrak{a}_+(E)$	set of positive self-adjoint automorphisms of E: 21.17
$\tilde{G}_u$, $\mathfrak{g}_u$, $\mathfrak{g}$, c_u	$\tilde{G}_u$ a simply connected compact semi-simple Lie group; $\mathfrak{g}_u = \text{Lie}(\tilde{G}_u)$; $\mathfrak{g} = (\mathfrak{g}_u)_{(C)}$; c_u the conjugation of $\mathfrak{g}$ for which $\mathfrak{g}_u$ is the set of fixed vectors: 21.18
$\tilde{G}$	simply connected complex Lie group with Lie algebra $\mathfrak{g}$: 21.18
c_0	conjugation of $\mathfrak{g}$ which commutes with c_u: 21.18
$\mathfrak{k}_0$, $i\mathfrak{p}_0$	real vector subspaces of $\mathfrak{g}_u$ on which $c_0(\mathbf{x}) = \mathbf{x}$ and $c_0(\mathbf{x}) = -\mathbf{x}$, respectively: 21.18
$\mathfrak{g}_0$	subalgebra of invariants of c_0: 21.18
$\tilde{P}$	image of $i\mathfrak{g}_u$ under the mapping $i\mathbf{u} \mapsto \exp_{\tilde{G}}(i\mathbf{u})$: 21.18
G_0, K_0, P_0	G_0 the Lie subgroup of $\tilde{G}_{\vert\mathbb{R}}$ consisting of the fixed points of σ such that $\sigma_* = c_0$; $K_0 = G_0 \cap \tilde{G}_u$; $P_0 = G_0 \cap \tilde{P}$: 21.18
G_1	$\tilde{G}_0/D$, a group locally isomorphic to $\tilde{G}_0$: 21.18
K_1, P_1	$K_1 = \tilde{K}_0/D$; P_1 = image of $\mathfrak{p}_0$ under $\exp_{G_1}$: 21.18
G_2	$\tilde{G}_u/(C \cap G_0)$, C the centre of $\tilde{G}_u$: 21.18
K_2	$K_0/(C \cap G_0)$: 21.18
K_2'	subgroup of fixed points of σ_2, the automorphism of G_2 obtained from σ on passing to the quotient: 21.18
P_2	image of $i\mathfrak{p}_0$ under $\exp_{G_2}$: 21.18
$\alpha \prec \beta$	lexicographic ordering: 21.20
$\mathfrak{a}_0$	maximal commutative subalgebra of $\mathfrak{p}_0$: 21.21
$\mathfrak{t}$	maximal commutative subalgebra of $\mathfrak{g}_u$ containing $\mathfrak{a}_0$: 21.21
$\mathbf{S}'$	subset of $\mathbf{S}$ consisting of the roots which vanish on $i\mathfrak{a}_0$: 21.21

$\mathbf{S}''_+$ — subset of $\mathbf{S}'' = \mathbf{S} - \mathbf{S}$ consisting of the α such that $\alpha(\mathbf{z}_0) > 0$: 21.21

$\mathfrak{n}, \mathfrak{n}_0$ — $\mathfrak{n} = \bigoplus_{\alpha \in \mathbf{S}^+} \mathfrak{g}_\alpha$, $\mathfrak{n} = \mathfrak{n}_0 \cap \mathfrak{g}_0$: 21.21

$\mathfrak{T}_k$ — Lie algebra of matrices (x_{hj}) such that $x_{hj} = 0$ for $j + k > h$: 21.21

CONTENTS

CHAPTER XXI

COMPACT LIE GROUPS AND SEMISIMPLE LIE GROUPS

It is rarely the case in mathematics that one can describe explicitly *all* the objects endowed with a structure that is characterized by a few simple axioms. A classical (and elementary) example is that of *finite commutative groups* (A.26.4). By contrast, in spite of more than a century of effort and an enormous accumulation of results, mathematics is still very far from being able to describe all *noncommutative finite groups*, even when supplementary restrictions (such as simplicity or nilpotency) are imposed.

It is therefore all the more remarkable that, in the theory of Lie groups, all the *compact simply connected* Lie groups are explicitly known, and that, starting from these groups, the structure of compact connected Lie groups is reduced to a simple problem in the theory of finitely generated commutative groups ((16.30.2) and (21.6.9)). The compact simply connected Lie groups are finite products of groups that are either the universal covering groups of the "classical groups" $\mathbf{SO}(n)$, $\mathbf{SU}(n)$, and $\mathbf{U}(n, \mathbf{H})$ (16.11) (and therefore depend on an integral parameter) or the five "exceptional" groups, of dimensions 14, 52, 78, 133, and 248. We shall not get as far as this final result, but we shall develop the methods leading to it, up to the point where what remains to be done is an enumeration (by successive exclusion) of certain algebraic objects related to Euclidean geometry, subjected to very restrictive conditions of an arithmetic nature, which allow only a small number of possibilities (21.10.3) (see [79] or [85] for a complete account).

These methods are based in part on the elementary theory of Lie groups in Chapter XIX, and in part on a fundamental new idea, which dominates this chapter and the next, and whose importance in present-day mathematics cannot be overemphasized; the notion of a *linear representation* of a group. The first essential fact is that where *compact* groups are concerned (whether they are Lie groups or not) we may restrict our attention to *finite-dimensional* linear representations (21.2.3). The second unexpected phenomenon is that where compact connected Lie groups are concerned, everything rests on the explicit knowledge of the representations of only *two* types of groups: the tori $\mathbf{T}^n$ and the group SU(2) (**21.9**). Roughly speaking, these are the "building blocks" with which we can "construct" all the other compact connected Lie groups and obtain not only their explicit structure but also an enumeration of *all* their linear representations (**21.15.5**).

The interest attached to the compact connected Lie groups arises not only from the esthetic attractions of the theory, which is one of the most beautiful and most satisfying in the whole of mathematics, but also from the central position they occupy in the welter of modern theories. In the first place, they are closely related to a capital notion in the theory of Lie groups, namely that of a *semisimple group* (compact or not), and in fact it turns out that a knowledge of the compact semisimple groups determines all the others (**21.18**). Since the time of F. Klein it has been recognized that classical "geometry" is essentially the study of certain semisimple groups; and E. Cartan, in his development of the notions of fiber bundle and connection, showed that these groups play an equally important role in differential geometry (see Chapter XX). From then on, their influence has spread into differential topology and homological algebra. We shall see in Chapter XXII how—again following E. Cartan—it has been realized over the last twenty-five years that the study of representations of semisimple groups (but now on infinite-dimensional spaces) is fundamental in many questions of analysis, not to speak of applications to quantum mechanics. But the most unexpected turn has been the invasion of the theory of semisimple groups into regions that appear completely foreign: "abstract" algebraic geometry, number theory, and the theory of finite groups. It has been known since the work of S. Lie and E. Cartan that semisimple groups are *algebraic* (that is, they can be defined by polynomial equations); but it is only since 1950 that it has come to be realized that this is no accidental fact, but rather that the theory of semisimple groups has *two faces* of equal importance: the analytic aspect, which gave birth to the theory, and the purely algebraic aspect, which appears when one considers a ground field other than **R** or **C**. We have not, unfortunately, been able to take account of this second aspect; here we can only remark that its repercussions are increasingly numerous, and refer the reader to the works [80], [81], [74], [77], and [78] in the bibliography.

1. CONTINUOUS UNITARY REPRESENTATIONS OF LOCALLY COMPACT GROUPS

(**21.1.1**) Let G be a topological group, E a Hausdorff topological vector space over the field **C** of complex numbers. Generalizing the definition given in (**16.9.7**), we define a *continuous linear representation of* G *on* E to be a mapping $s \mapsto U(s)$ of G into the group **GL**(E) of automorphisms of the *topological vector space* E, which satisfies the following conditions:

(a) $U(st) = U(s)U(t)$ for all $s, t \in$ G;
(b) for each $x \in$ E, the mapping $s \mapsto U(s) \cdot x$ of G into E is *continuous*.

It follows from (a) that $U(e) = 1_E$ (where e is the identity element of G) and that, for all $s \in$ G,

(21.1.1.1) $$U(s^{-1}) = U(s)^{-1}.$$

If E is of *finite* dimension d, the representation U is said to be of *dimension* (or *degree*) d, and we sometimes write $d = \dim U$.

The mapping U_0 that sends each $s \in$ G to the identity automorphism 1_E is a continuous linear representation of G on E, called the *trivial representation.*

A vector subspace F of E is said to be *stable* under a continuous linear representation U of G on E if $U(s)(F) \subset F$ for all $s \in$ G; in that case, the mapping $s \mapsto U(s) | F$ is a continuous linear representation of G on F, called the *subrepresentation* of U corresponding to F.

A continuous linear representation U of G on E is said to be *irreducible* (or *topologically irreducible*) if the only *closed* vector subspaces F of E that are stable under U are $\{0\}$ and E. For each $x \neq 0$ in E, the set $\{U(s) \cdot x : s \in G\}$ is then *total* in E (**12.13**).

(21.1.2) In this chapter and the next, we shall be concerned especially with the case where E is a *separable Hilbert space.* A *continuous unitary representation* of G on E is then a continuous linear representation U of G on E such that for each $s \in$ G the operator $U(s)$ is *unitary*, or in other words (**15.5**) is an automorphism of the Hilbert space structure of E. This means that the operators $U(s)$ satisfy conditions (a) and (b) of (**21.1.1**), together with the following condition:

(c) $(U(s) \cdot x \,|\, U(s) \cdot y) = (x \,|\, y)$ for all $s \in$ G and all $x, y \in$ E.

In particular, $U(s)$ is an *isometry* of E onto E, for all $s \in$ G, and we have

(21.1.2.1) $$U(s)^{-1} = (U(s))^*$$

for all $s \in$ G.

(21.1.3) (i) When E is *finite*-dimensional, condition (b) of (**21.1.1**) is equivalent to saying that $s \mapsto U(s)$ is a *continuous* mapping of G into the normed algebra $\mathscr{L}$(E) (relative to any norm that defines the topology of E); for it is equivalent to saying that if $(u_{jk}(s))$ is the matrix of $U(s)$ relative to some basis of E, then the functions u_{jk} are continuous on G. On the other hand, if E is a separable Hilbert space of infinite dimension and U is a continuous unitary representation of G on E, then U is not in general a continuous mapping of G into the normed algebra $\mathscr{L}(\mathrm{E})$ (Problem 3).

(ii) When E is finite-dimensional, a continuous linear representation U of G on E is not necessarily a continuous *unitary* representation relative to any scalar product **(6.2)** on E. For example, if $\mathrm{G} = \mathbf{R}$, the continuous linear representation

$$U : x \mapsto \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$$

of G on $\mathbf{C}^2$ is not unitary, relative to any scalar product on $\mathbf{C}^2$, because any unitary matrix is similar to a diagonal matrix **(15.11.14)** (cf. Section **21.18**, Problem 1).

(21.1.4) *Throughout the rest of this chapter we shall consider only separable metrizable locally compact groups, and as in Chapter XIV the phrases "locally compact group" and "compact group" will mean "separable metrizable locally compact group" and "metrizable compact group," respectively.*

Let G be a locally compact group, μ a *bounded complex measure* **(13.20)** on G, and U a continuous unitary representation of G on a separable Hilbert space E. For each pair of vectors x, y in E, the function $s \mapsto (U(s) \cdot x \mid y)$ is continuous and bounded on G, because $\|U(s) \cdot x\| = \|x\|$; it is therefore μ-integrable, and by **(13.20.5)** we have

(21.1.4.1)
$$\left| \int (U(s) \cdot x \mid y) \, d\mu(s) \right| \le \|\mu\| \cdot \|x\| \cdot \|y\|.$$

Since E may be identified with its dual, it follows that there exists a unique vector $U(\mu) \cdot x$ in E such that

$$\int (U(s) \cdot x \mid y) \, d\mu(s) = (U(\mu) \cdot x \mid y)$$

for all $y \in \mathrm{E}$, and this allows us to write **(13.10.6)**

(21.1.4.2)
$$U(\mu) \cdot x = \int (U(s) \cdot x) \, d\mu(s).$$

It is clear that this relation defines a continuous endomorphism $U(\mu)$ of E, since **(21.1.4.1)** implies that

(21.1.4.3)
$$\|U(\mu)\| \le \|\mu\|.$$

In particular, we have

(21.1.4.4) $$U(\varepsilon_s) = U(s)$$

for all $s \in G$.

The relation (21.1.4.2) is sometimes written in the abridged form

(21.1.4.5) $$U(\mu) = \int U(s)\, d\mu(s).$$

(21.1.5) We recall (15.4.9) that the set $M_{\mathbf{C}}^1(G)$ of bounded complex measures on G is an *involutory Banach algebra* over C, the multiplication being convolution of measures, and the involution $\mu \mapsto \check{\mu}$. When a left Haar measure β has been chosen on G, the normed space $L_{\mathbf{C}}^1(G)$ may be canonically identified with a closed vector subspace of $M_{\mathbf{C}}^1(G)$, by identifying the class $\tilde{f}$ of a β-integrable function f with the bounded measure $f \cdot \beta$, since $\|f \cdot \beta\| = N_1(f)$ (13.20.3). By the definition of the convolution of two functions in $\mathscr{L}_{\mathbf{C}}^1(G)$ (14.10.1), $L_{\mathbf{C}}^1(G)$ is a subalgebra of $M_{\mathbf{C}}^1(G)$ if we define the product of the classes of two functions $f, g \in \mathscr{L}_{\mathbf{C}}^1(G)$ to be the class of $f * g$. If in addition G is *unimodular* (14.3), $L_{\mathbf{C}}^1(G)$ is a two-sided ideal in $M_{\mathbf{C}}^1(G)$, and the transform of the measure $f \cdot \beta$ under the involution $\mu \mapsto \check{\mu}$ is $\check{f} \cdot \beta$ (14.3.4.2). We may therefore consider $L_{\mathbf{C}}^1(G)$ as an *involutory closed subalgebra* of $M_{\mathbf{C}}^1(G)$, the involution being that which transforms the class of f into the class of $\check{f}$.

We deduce from this that if G is *unimodular*, then for each *representation* (15.5) V of the involutory Banach algebra $L_{\mathbf{C}}^1(G)$ on a Hilbert space E, we have

(21.1.5.1) $$\|V(\tilde{f})\| \leqq N_1(f)$$

for all $f \in \mathscr{L}_{\mathbf{C}}^1(G)$. For if G is discrete, this is just (15.5.7) because the identity element ε_e of $M_{\mathbf{C}}^1(G)$ then belongs to $L_{\mathbf{C}}^1(G)$. If G is not discrete, it is immediately seen that V may be extended to a representation on E of the involutory Banach subalgebra $A = L_{\mathbf{C}}^1(G) \oplus \mathbf{C}\varepsilon_e$ of $M_{\mathbf{C}}^1(G)$ by putting $V(f \cdot \beta + \lambda\varepsilon_e) = V(\tilde{f}) + \lambda \cdot 1_E$, and (15.5.7) can then be applied to this algebra with identity element.

(21.1.6) *Under the assumptions of* (21.1.4), *the mapping* $\mu \mapsto U(\mu)$ *is a representation* (15.5) *of the involutory Banach algebra* $M_{\mathbf{C}}^1(G)$ *on the Hilbert space* E. *If in addition* G *is unimodular, the restriction of* $\mu \mapsto U(\mu)$ *to* $L_{\mathbf{C}}^1(G)$ *is nondegenerate.*

It follows immediately from (21.1.4.4) that $U(\varepsilon_e) = 1_E$. To prove the first assertion, it remains to show that $U(\mu * \nu) = U(\mu)U(\nu)$ and $U(\check{\mu}) = (U(\mu))^*$, where μ, ν are any two bounded measures on G. If x, y are any two vectors in E then by definition (14.5) we have

$$\begin{aligned}(U(\mu * \nu) \cdot x \mid y) &= \int (U(s) \cdot x \mid y)\, d(\mu * \nu)(s) \\ &= \iint (U(vw) \cdot x \mid y)\, d\mu(v)\, d\nu(w) \\ &= \iint (U(w) \cdot x \mid (U(v))^* \cdot y)\, d\mu(v)\, d\nu(w) \\ &= \int (U(\nu) \cdot x \mid (U(v))^* \cdot y)\, d\mu(v) \\ &= \int (U(v) \cdot (U(\nu) \cdot x) \mid y)\, d\mu(v) \\ &= (U(\mu)U(\nu) \cdot x \mid y)\end{aligned}$$

by virtue of the Lebesgue–Fubini theorem, and this proves the first relation. Next, using the fact that the operators $U(s)$ are unitary, we have

$$\begin{aligned}((U(\mu))^* \cdot x \mid y) &= \overline{(U(\mu) \cdot y \mid x)} \\ &= \int \overline{(U(s) \cdot y \mid x)\, d\mu(s)} \\ &= \int \overline{(U(s) \cdot y \mid x)}\, d\bar{\mu}(s) \\ &= \int (U(s^{-1}) \cdot x \mid y)\, d\bar{\mu}(s) \\ &= \int (U(t) \cdot x \mid y)\, d\check{\bar{\mu}}(t) \\ &= (U(\check{\bar{\mu}}) \cdot x \mid y)\end{aligned}$$

by the definition of the measure $\check{\bar{\mu}}$ (15.4.9), and this proves the second relation.

In particular, for each $s \in$ G and each bounded measure μ on G, we have

(21.1.6.1) $$U(\varepsilon_s * \mu) = U(s)U(\mu), \qquad U(\mu * \varepsilon_s) = U(\mu)U(s).$$

Let (V_n) be a decreasing sequence of neighborhoods of e in G, forming a fundamental system of neighborhoods of e. For each $s \in$ G and each n, let u_n