PREFACE

An American student approaching the higher parts of mathematics usually finds himself unfamiliar with most of the main facts of algebra, to say nothing of their proofs. Thus he has only a rudimentary knowledge of systems of linear equations, and he knows next to nothing about the subject of quadratic forms. Students in this condition, if they receive any algebraic instruction at all, are usually plunged into the detailed study of some special branch of algebra, such as the theory of equations or the theory of invariants, where their lack of real mastery of algebraic principles makes it almost inevitable that the work done should degenerate to the level of purely formal manipulations. It is the object of the present book to introduce the student to higher algebra in such a way that he shall, on the one hand, learn what is meant by a proof in algebra and acquaint himself with the proofs of the most fundamental facts, and, on the other, become familiar with many important results of algebra which are new to him.

The book being thus intended, not as a compendium, but really, as its title states, only as an *introduction* to higher algebra, the attempt has been made throughout to lay a sufficiently broad foundation to enable the reader to pursue his further studies intelligently, rather than to carry any single topic to logical completeness. No apology seems necessary for the omission of even such important subjects as Galois's Theory and a systematic treatment of invariants. A selection being necessary, those subjects have been chosen for treatment which have proved themselves of greatest importance in geometry and analysis, as well as in algebra, and the relations of the algebraic theories to geometry have been emphasized throughout. At the same time it must be borne in mind that the subject primarily treated is algebra, not analytic geometry, so that such geometric information as is given is necessarily of a fragmentary and somewhat accidental character.

No algebraic knowledge is presupposed beyond a familiarity with elementary algebra up to and including quadratic equations, and such a knowledge of determinants and the method of mathematical induction as may easily be acquired by a freshman in a week or two. Nevertheless, the book is not intended for wholly immature readers, but rather for students who have had two or three years' training in the elements of higher mathematics, particularly in analytic geometry and the calculus. In fact, a good elementary knowledge of analytic geometry is indispensable.

The exercises at the ends of the sections form an essential part of the book, not merely in giving the reader an opportunity to think for himself on the subjects treated, but also, in many cases, by supplying him with at least the outlines of important additional theories. As illustrations of this we may mention Sylvester's Law of Nullity (page 80), orthogonal transformations (page 154 and page 173), and the theory of the invariants of the biquadratic binary form (page 260).

On a first reading of Chapters I–VII, it may be found desirable to omit some or all of sections 10, 11, 18, 19, 20, 25, 27, 34, 35. The reader may then either take up the subject of quadratic forms (Chapters VIII–XIII), or, if he prefer, he may pass directly to the more general questions treated in Chapters XIV–XIX.

The chapters on Elementary Divisors (XX–XXII) form decidedly the most advanced and special portion of the book. A person wishing to read them without reading the rest of the book should first acquaint himself with the contents of sections 19 (omitting Theorem 1), 21–25, 36, 42, 43.

In a work of this kind, it has not seemed advisable to give many bibliographical references, nor would an acknowledgement at this point of the sources from which the material has been taken be feasible. The work of two mathematicians, however, Kronecker and Frobenius, has been of such decisive influence on the character of the book that it is fitting that their names receive special mention here. The author would also acknowledge his indebtedness to his colleague, Professor Osgood, for suggestions and criticisms relating to Chapters XIV–XVI.

This book has grown out of courses of lectures which have been delivered by the author at Harvard University during the last ten years. His thanks are due to Mr. Duval, one of his former pupils, without whose assistance the book would probably never have been written.

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INTRODUCTION TO HIGHER ALGEBRA

CHAPTER I

POLYNOMIALS AND THEIR MOST FUNDAMENTAL PROPERTIES

1. Polynomials in One Variable. By an integral rational function of x, or, as we shall say for brevity, a polynomial in x, is meant a function of x determined by an expression of the form

$$(1) \qquad c_1x^{\alpha_1} + c_2x^{\alpha_2} + \cdots + c_kx^{\alpha_k},$$

where the α's are integers positive or zero, while the c's are any constants, real or imaginary. We may without loss of generality assume that no two of the α's are equal. This being the case, the expressions $c_ix^{\alpha_i}$ are called the *terms* of the polynomial, c_i is called the *coefficient* of this term, and α_i is called its *degree*. The highest degree of any term whose coefficient is not zero is called the *degree of the polynomial.*

It should be noticed that the conceptions just defined — terms, coefficients, degree — apply not to the polynomial itself, but to the particular *expression* (1) which we use to determine the polynomial, and it would be quite conceivable that one and the same function of x might be given by either one of two wholly different expressions of the form (1). We shall presently see (cf. Theorem 5 below) that this cannot be the case except for the obvious fact that we may insert in or remove from (1) any terms we please with zero coefficients.

By arranging the terms in (1) in the order of decreasing α's and supplying, if necessary, certain missing terms with zero coefficients, we may write the polynomial in the normal form

$$(2) \qquad a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n.$$

It should, however, constantly be borne in mind that a polynomial in this form is not necessarily of the nth degree; but will be of the nth degree when and only when $a_0 \neq 0$.

DEFINITION. *Two polynomials, $f_1(x)$ and $f_2(x)$, are said to be identically equal ($f_1 \equiv f_2$) if they are equal for all values of x. A polynomial $f(x)$ is said to vanish identically ($f \equiv 0$) if it vanishes for all values of x.*

We learn in elementary algebra how to add, subtract, and multiply * polynomials; that is, when two polynomials $f_1(x)$ and $f_2(x)$ are given, to form new polynomials equal to the sum, difference, and product of these two.

THEOREM 1. *If the polynomial*

$$f(x) \equiv a_0 x^n + a_1 x^{n-1} + \cdots + a_n$$

vanishes when $x = \alpha$, there exists another polynomial

$$\phi_1(x) \equiv a_0 x^{n-1} + a_1' x^{n-2} + \cdots + a_{n-1}',$$

such that

$$f(x) \equiv (x - \alpha)\phi_1(x).$$

For since by hypothesis $f(\alpha) = 0$, we have

$$f(x) \equiv f(x) - f(\alpha) \equiv a_0(x^n - \alpha^n) + a_1(x^{n-1} - \alpha^{n-1}) + \cdots + a_{n-1}(x - \alpha).$$

Now by the rule of elementary algebra for multiplying together two polynomials we have

$$x^k - \alpha^k \equiv (x - \alpha)(x^{k-1} + \alpha x^{k-2} + \cdots + \alpha^{k-1}).$$

Hence

$$f(x) \equiv (x - \alpha)[a_0(x^{n-1} + \alpha x^{n-2} + \cdots + \alpha^{n-1}) + a_1(x^{n-2} + \alpha x^{n-3} + \cdots + \alpha^{n-2}) + \cdots + a_{n-1}].$$

If we take as $\phi_1(x)$ the polynomial in brackets, our theorem is proved.

Suppose now that β is another value of x distinct from α for which $f(x)$ is zero. Then

$$f(\beta) = (\beta - \alpha)\phi_1(\beta) = 0;$$

* The question of division is somewhat more complicated and will be considered in § 63.

and since $\beta-\alpha\neq 0$, $\phi_1(\beta)=0$. We can therefore apply the theorem just proved to the polynomial $\phi_1(x)$, thus getting a new polynomial

$$\phi_2(x)\equiv a_0x^{n-2}+a_1''x^{n-3}+\cdots+a_{n-2}'',$$

such that

$$\phi_1(x)\equiv(x-\beta)\phi_2(x),$$

and therefore

$$f(x)\equiv(x-\alpha)(x-\beta)\phi_2(x).$$

Proceeding in this way, we get the following general result:

THEOREM 2. *If $\alpha_1, \alpha_2, \cdots \alpha_k$ are k distinct constants, and if*

$$f(x)\equiv a_0x^n+a_1x^{n-1}+\cdots+a_n \qquad (n\geqq k),$$

and

$$f(\alpha_1)=f(\alpha_2)=\cdots=f(\alpha_k)=0,$$

then

$$f(x)\equiv(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_k)\phi(x),$$

where

$$\phi(x)\equiv a_0x^{n-k}+b_1x^{n-k-1}+\cdots+b_{n-k}.$$

Applying this theorem in particular to the case $n=k$, we see that if the polynomial $f(x)$ vanishes for n distinct values $\alpha_1, \alpha_2, \cdots \alpha_n$ of x, then

$$f(x)\equiv a_0(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n).$$

Accordingly, if $a_0\neq 0$, there can be no value of x other than $\alpha_1, \cdots \alpha_n$ for which $f(x)=0$. We have thus proved

THEOREM 3. *A polynomial of the nth degree in x cannot vanish for more than n distinct values of x.*

Since the only polynomials which have no degree are those all of whose coefficients are zero, and since such polynomials obviously vanish identically, we get the fundamental result:

THEOREM 4. *A necessary and sufficient condition that a polynomial in x vanish identically is that all its coefficients be zero.*

Since two polynomials in x are identically equal when and only when their difference vanishes identically, we have

THEOREM 5. *A necessary and sufficient condition that two polynomials in x be identically equal is that they have the same coefficients.*

This theorem shows, as was said above, that the terms, coefficients, and degree of a polynomial depend merely on the polynomial itself, not on the special way in which it is expressed.

2. Polynomials in More than One Variable. A function of (x, y) is called a polynomial if it is given by an expression of the form

$$c_1x^{\alpha_1}y^{\beta_1} + c_2x^{\alpha_2}y^{\beta_2} + \cdots + c_kx^{\alpha_k}y^{\beta_k},$$

where the α's and β's are integers positive or zero.

More generally, a function of $(x_1, x_2, \cdots x_n)$ is called a polynomial if it is determined by an expression of the form

$$(1) \qquad c_1x_1^{\alpha_1}x_2^{\beta_1}\cdots x_n^{\nu_1} + c_2x_1^{\alpha_2}x_2^{\beta_2}\cdots x_n^{\nu_2} + \cdots + c_kx_1^{\alpha_k}x_2^{\beta_k}\cdots x_n^{\nu_k},$$

where the α's, β's, $\cdots$ ν's are integers positive or zero.

Here we may assume without loss of generality that in no two terms are the exponents of the various x's the same; that is, that if

$$\alpha_i = \alpha_j,\ \beta_i = \beta_j,\ \cdots\ \mu_i = \mu_j,$$

then

$$\nu_i \neq \nu_j.$$

This assumption being made, $c_ix_1^{\alpha_i}x_2^{\beta_i}\cdots x_n^{\nu_i}$ is called a *term* of the polynomial, c_i its *coefficient*, α_i the *degree of the term in* x_1, β_i in x_2, etc., and $\alpha_i + \beta_i + \cdots + \nu_i$ the *total degree*, or simply the degree, of the term. The highest degree in x_i of any term in the polynomial whose coefficient is not zero is called the *degree of the polynomial in* x_i, and the highest total degree of any term whose coefficient is not zero is called the *degree of the polynomial.*

Here, as in § 1, the conceptions just defined apply for the present not to the function itself but to the special method of representing it by an expression of the form (1). We shall see presently, however, that this method is unique.

Before going farther, we note explicitly that *according to the definition we have given, a polynomial all of whose coefficients are zero has no degree.*

When we speak of a polynomial in n variables, we do not necessarily mean that all n variables are actually present. One or more of them may have the exponent zero in every term, and hence not appear at all. Thus a polynomial in one variable, or even a constant, may be regarded as a special case of a polynomial in any larger number of variables.

A polynomial all of whose terms are of the same degree is said to be *homogeneous.* Such polynomials we will speak of as *forms*,*

* The e is diversity of usage here. Some writers, following Kronecker, apply the term *form* to all polynomials. On the other hand, homogeneous polynomials are often spoken of as *quantics* by English writers.

distinguishing between *binary*, *ternary*, *quaternary*, and in general, *n-ary* forms according to the number of variables involved, binary forms involving two, ternary three, etc.

Another method of classifying forms is according to their degree. We speak here of linear forms, quadratic forms, cubic forms, etc., according as the degree is 1, 2, 3, etc. We will, however, agree that a polynomial all of whose coefficients are zero may also be spoken of indifferently as a linear form, quadratic form, cubic form, etc., in spite of the fact that it has no degree.

If all the coefficients of a polynomial are real, it is called a *real polynomial* even though, in the course of our work, we attribute imaginary values to the variables.

It is frequently convenient to have a polynomial in more than one variable arranged according to the descending powers of some one of the variables. Thus a normal form in which we may write a polynomial in n variables is

$$\phi_0(x_2, \cdots x_n)\,x_1^m + \phi_1(x_2, \cdots x_n)\,x_1^{m-1} + \cdots + \phi_m(x_2, \cdots x_n),$$

the ϕ's being polynomials in the $n-1$ variables $(x_2, \cdots x_n)$.

We learn in elementary algebra how to add, subtract, and multiply polynomials, getting as the result new polynomials.

DEFINITION. *Two polynomials in any number of variables are said to be identically equal if they are equal for all values of the variables. A polynomial is said to vanish identically if it vanishes for all values of the variables.*

THEOREM 1. *A necessary and sufficient condition that a polynomial in any number of variables vanish identically is that all its coefficients be zero.*

That this is a sufficient condition is at once obvious. To prove that it is a necessary condition we use the method of mathematical induction. Since we know that the theorem is true in the case of one variable (Theorem 4, § 1), the theorem will be completely proved if we can show that if it is true for a certain number $n-1$ of variables, it is true for n variables.

Suppose, then, that

$$f(x_1, \cdots x_n) \equiv \phi_0(x_2, \cdots x_n)x_1^m + \phi_1(x_2, \cdots x_n)x_1^{m-1} + \cdots + \phi_m(x_2, \cdots x_n)$$

vanishes identically. If we assign to $(x_2, \cdots x_n)$ any fixed values $(x_2', \cdots x_n')$, f becomes a polynomial in x_1 alone, which, by hypothesis, vanishes for all values of x_1. Hence its coefficients must, by Theorem 4, § 1, all be zero:

$$\phi_i(x_2', \cdots x_n') = 0 \qquad (i = 0, 1, \cdots m)$$

That is, the polynomials $\phi_0, \phi_1, \cdots \phi_m$ vanish for all values of the variables, since $(x_2', \cdots x_n')$ was *any* set of values. Accordingly, by the assumption we have made that our theorem is true for polynomials in $n-1$ variables, all the coefficients of all the polynomials $\phi_0, \phi_1, \cdots \phi_m$ are zero. These, however, are simply the coefficients of f. Thus our theorem is proved.

Since two polynomials are identically equal when and only when their difference is identically zero, we infer now at once the further theorem:

THEOREM 2. *A necessary and sufficient condition that two polynomials be identically equal is that the coefficients of their corresponding terms be equal.*

We come next to

THEOREM 3. *If f_1 and f_2 are polynomials in any number of variables of degrees m_1 and m_2 respectively, the product $f_1 f_2$ will be of degree $m_1 + m_2$.*

This theorem is obviously true in the case of polynomials in one variable. If, then, assuming it to be true for polynomials in $n-1$ variables we can prove it to be true for polynomials in n variables, the proof of our theorem by the method of mathematical induction will be complete.

Let us look first at the special case in which both polynomials are homogeneous. Here every term we get by multiplying them together by the method of elementary algebra is of degree $m_1 + m_2$. Our theorem will therefore be proved if we can show that there is at least one term in the product whose coefficient is not zero. For this purpose, let us arrange the two polynomials f_1 and f_2 according to descending powers of x_1,

$$f_1(x_1, \cdots x_n) \equiv \phi_0'(x_2, \cdots x_n)x_1^{k_1} + \phi_1'(x_2, \cdots x_n)x_1^{k_1-1} + \cdots,$$

$$f_2(x_1, \cdots x_n) \equiv \phi_0''(x_2, \cdots x_n)x_1^{k_2} + \phi_1''(x_2, \cdots x_n)x_1^{k_2-1} + \cdots.$$

Here we may assume that neither ϕ_0' nor ϕ_0'' vanishes identically. Since f_1 and f_2 are homogeneous, ϕ_0' and ϕ_0'' will also be homogeneous of degrees $m_1 - k_1$ and $m_2 - k_2$ respectively. In the product $f_1 f_2$ the terms of highest degree in x_1 will be those in the product

$$\phi_0'(x_2, \cdots x_n)\phi_0''(x_2, \cdots x_n)x_1^{k_1+k_2},$$

and since we assume our theorem to hold for polynomials in $n-1$ variables, $\phi_0'\phi_0''$ will be a polynomial of degree $m_1 + m_2 - k_1 - k_2$. Any term in this product whose coefficient is not zero gives us when multiplied by $x_1^{k_1+k_2}$ a term of the product $f_1 f_2$ of degree $m_1 + m_2$ whose coefficient is not zero. Thus our theorem is proved for the case of homogeneous polynomials.

Let us now, in the general case, write f_1 and f_2 in the forms

$$f_1(x_1, \cdots x_n) \equiv \phi_{m_1}'(x_1, \cdots x_n) + \phi_{m_1-1}'(x_1, \cdots x_n) + \cdots,$$

$$f_2(x_1, \cdots x_n) \equiv \phi_{m_2}''(x_1, \cdots x_n) + \phi_{m_2-1}''(x_1, \cdots x_n) + \cdots,$$

where ϕ_i' and ϕ_j'' are homogeneous polynomials which are either of degrees i and j respectively, or which vanish identically. Since, by hypothesis, f_1 and f_2 are of degrees m_1 and m_2 respectively, ϕ_{m_1}' and ϕ_{m_2}'' will not vanish identically, but will be of degrees m_1 and m_2.

The terms of highest degree in the product $f_1 f_2$ will therefore be the terms of the product $\phi_{m_1}' \phi_{m_2}''$, and this being a product of homogeneous polynomials comes under the case just treated and is therefore of degree $m_1 + m_2$. The same is therefore true of the product $f_1 f_2$, and our theorem is proved.

By a successive application of this theorem we infer

COROLLARY. *If k polynomials are of degrees $m_1, m_2, \cdots m_k$ respectively, their product is of degree $m_1 + m_2 + \cdots + m_k$.*

We mention further, on account of their great importance, the two rather obvious results :

THEOREM 4. *If the product of two or more polynomials is identically zero, at least one of the factors must be identically zero.*

For if none of them were identically zero, they would all have definite degrees, and therefore their product would, by Theorem 3, have a definite degree, and would therefore not vanish identically.

It is from this theorem that we draw our justification for cancelling out from an identity a factor which we know to be not identically zero.

THEOREM 5. *If $f(x_1, \cdots x_n)$ is a polynomial which is not identically zero, and if $\phi(x_1, \cdots x_n)$ vanishes at all points where f does not vanish, then ϕ vanishes identically.*

This follows from Theorem 4 when we notice that $f\phi \equiv 0$.

EXERCISES

1. If f and ϕ are polynomials in any number of variables, what can be inferred from the identity $f^2 \equiv \phi^2$ concerning the relation between the polynomials f and ϕ?

2. If f_1 and f_2 are polynomials in $(x_1, \cdots x_n)$ which are of degrees m_1 and m_2 respectively in x_1, prove that their product is of degree $m_1 + m_2$ in x_1.

3. Geometric Interpretations. In dealing with functions of a single real variable, the different values which the variable may take on may be represented geometrically by the points of a line; it being understood that when we speak of a point x we mean the point which is situated on the line at a distance of x units (to the right or left according as x is positive or negative) from a certain fixed origin O, on the line. Similarly, in the case of functions of two real variables, the sets of values of the variables may be pictured geometrically by the points of a plane, and in the case of three real variables, by the points of space; the set of values represented by a point being, in each case, the *rectangular* coördinates of that point. When we come to functions of four or more variables, however, this geometric representation is impossible.

The complex variable $x = \xi + \eta i$ depends on the two independent real variables ξ and η in such a way that to every pair of real values (ξ, η) there corresponds one and only one value of x. The different values which a single complex variable may take on may, therefore, be represented by the points of a plane in which (ξ, η) are used as cartesian coördinates. In dealing with functions of more than one complex variable, however, this geometric representation is impossible, since even two complex variables $x = \xi + \eta i$, $y = \xi_1 + \eta_1 i$ are equivalent to four real variables $(\xi, \eta, \xi_1, \eta_1)$.

By the *neighborhood of a point* $x = a$ we mean that part of the line between the points $x = a - \alpha$ and $x = a + \alpha$ (α being an arbitrary positive constant, large or small), or what is the same thing, all points whose coördinates x satisfy the inequality $|x - a| < \alpha$.*

* We use the symbol $|Z|$ to denote the absolute value of Z, *i.e.* the numerical value of Z if Z is real, the modulus of Z if Z is imaginary.

Similarly, by the neighborhood of a point (a, b) in a plane, we shall mean all points whose coördinates (x, y) satisfy the inequalities

$$|x-a|<\alpha, \quad |y-b|<\beta,$$

where α and β are positive constants. This neighborhood thus consists of the interior of a rectangle of which (a, b) is the center and whose sides are parallel to the coördinate axes.

By the neighborhood of a point (a, b, c) in space we mean all points whose coördinates (x, y, z) satisfy the inequalities

$$|x-a|<\alpha, \quad |y-b|<\beta, \quad |z-c|<\gamma.$$

In all these cases it will be noticed that the neighborhood may be large or small according to the choice of the constants α, β, γ.

If we are dealing with a single complex variable $x=\xi+\eta i$, we understand by the neighborhood of a point a all points in the plane of complex quantities whose complex coördinate x satisfies the inequality $|x-a|<\alpha$, α being as before a real positive constant. Since $|x-a|$ is equal to the distance between x and a, the neighborhood of a now consists of the interior of a circle of radius α described about a as center.

It is found convenient to extend the geometric terminology we have here introduced to the case of any number of real or complex variables. Thus if we are dealing with n independent variables $(x_1, x_2, \cdots x_n)$, we speak of any particular set of values of these variables as a *point in space of n dimensions.* Here we have to distinguish between *real points*, that is sets of values of the x's which are all real, and *imaginary points* in which this is not the case. In using these terms we do not propose even to raise the question whether in any geometric sense there is such a thing as space of more than three dimensions. We merely use these terms in a wholly conventional algebraic sense because on the one hand they have the advantage of conciseness over the ordinary algebraic terms, and on the other hand, by calling up in our minds the geometric pictures of three dimensions or less, this terminology is often suggestive of new relations which might otherwise not present themselves to us so readily.

By the neighborhood of the point $(a_1, a_2, \cdots a_n)$ we understand all points which satisfy the inequalities

$$|x_1 - a_1| < \alpha_1, \quad |x_2 - a_2| < \alpha_2, \cdots |x_n - a_n| < \alpha_n,$$

where $\alpha_1, \alpha_2, \cdots \alpha_n$ are real positive constants.

If, in particular, $(a_1, a_2, \cdots a_n)$ is a real point, we may speak of the real neighborhood of this point, meaning thereby all real points $(x_1, x_2, \cdots x_n)$ which satisfy the above inequalities.

As an illustration of the use to which the conception of the neighborhood of a point can be put in algebra, we will prove the following important theorem:

THEOREM 1. *A necessary and sufficient condition that a polynomial $f(x_1, \cdots x_n)$ vanish identically is that it vanish throughout the neighborhood of a point $(a_1, \cdots a_n)$.*

That this is a necessary condition is obvious. To prove that it is sufficient we begin with the case $n = 1$.

Suppose then that $f(x)$ vanishes throughout a certain neighborhood of the point $x = a$. If $f(x)$ did not vanish identically, it would be of some definite degree, say k, and therefore could not vanish at more than k points (cf. Theorem 3, § 1). This, however, is not the case, since it vanishes at an infinite number of points, namely all points in the neighborhood of $x = a$. Thus our theorem is proved in the case $n = 1$.

Turning now to the case $n = 2$, let

$$f(x, y) \equiv \phi_0(y)x^k + \phi_1(y)x^{k-1} + \cdots + \phi_k(y)$$

be a polynomial which vanishes throughout a certain neighborhood of the point (a, b), say when

$$|x - a| < \alpha, \quad |y - b| < \beta.$$

Let y_0 be any constant satisfying the inequality

$$|y_0 - b| < \beta.$$

Then $f(x, y_0)$ is a polynominal in x alone which vanishes whenever $|x - a| < \alpha$. Hence, by the case $n = 1$ of our theorem, $f(x, y_0) \equiv 0$. That is,

$$\phi_0(y_0) = \phi_1(y_0) = \cdots = \phi_k(y_0) = 0.$$