Multiple Integrals in the Calculus of Variations

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Preface

The principal theme of this book is "the existence and differentiability of the solutions of variational problems involving multiple integrals." We shall discuss the corresponding questions for single integrals only very briefly since these have been discussed adequately in every other book on the calculus of variations. Moreover, applications to engineering, physics, etc., are not discussed at all; however, we do discuss *mathematical* applications to such subjects as the theory of harmonic integrals and the so-called "$\bar{\partial}$-Neumann" problem (see Chapters 7 and 8). Since the plan of the book is described in Section 1.2 below we shall merely make a few observations here.

In order to study the questions mentioned above it is necessary to use some very elementary theorems about convex functions and operators on Banach and Hilbert spaces and some special function spaces, now known as "SOBOLEV spaces". However, most of the facts which we use concerning these spaces were known before the war when a different terminology was used (see CALKIN and MORREY [5]); but we have included some powerful new results due to CALDERON in our exposition in Chapter 3. The definitions of these spaces and some of the proofs have been made simpler by using the most elementary ideas of distribution theory; however, almost no other use has been made of that theory and no knowledge of that theory is required in order to read this book. Of course we have found it necessary to develop the theory of linear elliptic systems at some length in order to present our desired differentiability results. We found it particularly essential to consider "weak solutions" of such systems in which we were often forced to allow discontinuous coefficients; in this connection, we include an exposition of the DE GIORGI—NASH—MOSER results. And we include in Chapter 6 a proof of the analyticity of the solutions (on the interior and at the boundary) of the most general non-linear analytic elliptic system with general regular (as in AGMON, DOUGLIS, and NIRENBERG) boundary conditions. But we confine ourselves to functions which are analytic, of class C^∞, of class C^n_μ or C^n (see § 1.2), or in some Sobolev space H^m_p with m an integer ≥ 0 (except in Chapter 9). These latter spaces have been defined for all real m in a domain (or manifold) or on its boundary and have been used by many authors in their studies of linear systems. We have not included a study of these spaces since (i) this book is already sufficiently long, (ii) we took no part in this development, and (iii) these spaces are adequately discussed in other *books* (see A. FRIEDMAN [2], HORMANDER [1], LIONS [2]) as well as in many papers (see § 1.8 and papers by LIONS and MAGENES).

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CHARLES B. MORREY, JR.

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Chapter 1

Introduction

1.1. Introductory remarks

The principal theme of these lectures is "the existence and differentiability of the solutions of variational problems involving multiple integrals." I shall discuss the corresponding questions for single integrals only very briefly since these have been adequately discussed in every book on the calculus of variations (see, for instance, AKHIEZER [1], BLISS [1], BOLZA [1], CARATHEODORY [2], FUNK [1], PARS [1]. Moreover, I shall not discuss applications to engineering, physics, etc., at all, although I shall mention some *mathematical* applications.

In general, I shall consider integrals of the form

$$(1.1.1) \qquad I(z,G) = \int_G f[x, z(x), \nabla z(x)]\, dx$$

where G is a domain,

$$(1.1.2) \qquad x = (x^1, \ldots, x^\nu), \; z = (z^1, \ldots, z^N), \; dx = dx^1 \ldots dx^\nu.$$

$z(x)$ is a vector function, ∇z denotes its gradient which is the set of functions $\{z^i_{,\alpha}\}$, where $z^i_{,\alpha}$ denotes $\partial z^i/\partial x^\alpha$, and $f(x, z, p)\,(p = \{p^i_\alpha\})$ is generally assumed continuous in all its arguments. The integrals

$$\int_a^b \sqrt{1 + (dz/dx)^2}\, dx \quad \text{and} \quad \iint_G \left[\left(\frac{\partial z}{\partial x^1}\right)^2 + \left(\frac{\partial z}{\partial x^2}\right)^2\right] dx^1\, dx^2$$

are familiar examples of integrals of the form (1.1.1.) in which $N = 1$ in both cases, $\nu = 1$ in the first case, $\nu = 2$ in the second case and the corresponding functions f are defined respectively by

$$f(x, z, p) = \sqrt{1 + p^2}, \qquad f(x, z, p) = p_1^2 + p_2^2$$

where we have omitted the superscripts on z and p since $N = 1$. The second integral is a special case of the *Dirichlet integral* which is defined in general by

$$(1.1.3) \qquad D(z, G) = \int_G |\nabla z|^2\, dx, \; f(x, z, p) = |p|^2 = \sum_{i,\alpha} (p^i_\alpha)^2.$$

Another example is the *area integral*

$$(1.1.4) \qquad A(z, G) = \iint_G \sqrt{\left[\frac{\partial(z^2, z^3)}{\partial(x^1, x^2)}\right]^2 + \left[\frac{\partial(z^3, z^1)}{\partial(x^1, x^2)}\right]^2 + \left[\frac{\partial(z^1, z^2)}{\partial(x^1, x^2)}\right]^2}\, dx^1\, dx^2$$

which gives the area of the surface

$$z^i = z^i(x^1, x^2), \quad (x^1, x^2) \in G, \quad i = 1, 2, 3. \tag{1.1.5}$$

It is to be noticed that the area integral has the special property that it is invariant under diffeomorphisms (1 — 1 differentiable mappings, etc.) of the domain G onto other domains. This is the first example of an *integral in parametric form*. I shall discuss such integrals later (in Chapters 9 and 10).

I shall also discuss briefly integrals like that in (1.1.1) but involving derivatives of higher order. And, of course, the variational *method* has been used in problems which involve a "functional" not at all like the integral in (1.1.1); as for example in proving the Riemann mapping theorem where one minimizes sup $|f(z)|$ among all schlicht functions $f(z)$ defined on the given simply connected region G for which $f(z_0) = 0$ and $f'(z_0) = 1$ at some given point z_0 in G.

We shall consider only problems in which the domain G is fixed; variations in G may be taken care of by transformations of coordinates. We shall usually consider problems involving fixed boundary values; we shall discuss other problems but will not derive the *transversality* conditions for such problems.

1.2. The plan of the book: notation

In this chapter we attempt to present an overall view of the principal theme of the book as stated at the beginning of the preceding section. However, we do not include a discussion of integrals in parametric form; these are discussed at some length in Chapters 9 and 10. The material in this book is not presented in its logical order. A possible logical order would be § 1.1 — 1.5, Chapter 2, Chapter 3, §§ 5.1 — 5.8, § 5.12, Chapter 6, §§ 1.6 — 1.9, §§ 4.1, 4.3, 4.4. Then the reader must skip back and forth as required among the material of § 1.10, 1.11, 4.2, 5.9, 5.10 and 5.11. Then the remainder of the book may be read substantially in order. Actually, Chapters 7 and 8 could be read immediately after § 5.8.

We begin by presenting background material including derivations, under restrictive hypotheses, of Euler's equations and the classical necessary conditions of Legendre and Weierstrass. Next, we include a brief and incomplete presentation of the classical so-called "sufficiency" conditions, including references to other works where a more complete presentation may be found.

The second half of this chapter presents a reasonably complete outline of the existence and differentiability theory for the solutions of variational problems. This begins with a brief discussion of the development of the direct methods and of the successively more general classes of "admissible" functions, culminating in the so-called "Sobolev spaces".

These are then defined and discussed briefly after which two theorems on lower-semicontinuity are presented. These are not the most general theorems possible but are selected for the simplicity of their proofs which, however, assume that the reader is willing to grant the truth of some well-known theorems on the Sobolev spaces. The relevant theorems about these spaces are proved in Chapter 3 and more general lower-semicontinuity and existence theorems are presented in Chapter 4.

In Section 1.10 the differentiability results are stated and some preliminary results are proved. In Section 1.11, an outline of the differentiability theory is presented. It is first shown that the solutions are "weak solutions" of the Euler equations. The theory of these non-linear equations is reduced to that of linear equations which, initially, may have discontinuous coefficients. The theory of these general linear equations is discussed in detail in Chapter 5. However, the higher order differentiability for the solutions of *systems* of Euler equations required the same methods as are used in studying systems of equations of higher order. Accordingly, we present in Chapter 6 many of the results in the two recent papers of AGMON, DOUGLIS, and NIRENBERG ([1], [2]) concerning the solutions and weak solutions of such systems. Both the L_p-estimates and the SCHAUDER-type estimates (concerning HÖLDER continuity) are presented. We have included sections in both Chapters 5 und 6 proving the analyticity, including analyticity at the boundary, of the solutions of both linear and non-linear analytic elliptic equations and systems; the most general "properly elliptic" systems with "complementing boundary conditions" (see § 6.1) are treated. The proof of analyticity in this generality is new. In Chapter 2 we present well-known facts about harmonic functions and generalized potentials and conclude with proofs of the CALDERON-ZYGMUND inequalities and of the maximum principle for the solutions of second order equations.

In Chapters 7 and 8, we present applications of the variational method to the HODGE theory of harmonic integrals and to the so-called $\bar{\partial}$-NEUMANN problem for exterior differential forms on strongly pseudo-convex complex analytic manifolds with boundary. In Chapter 9, we present a brief discussion of ν-dimensional parametric problems in general and then discuss the two dimensional Plateau problem in Euclidean space and on a Riemannian manifold. The chapter concludes with the author's simplified proof of the existence theorem of CESARI [4], DANSKIN, and SIGALOV [2] for the general two dimensional parametric problem and *some incomplete results concerning the differentiability of the solutions* of such problems. In Chapter 10, we present the author's simplification of the very important resent work of REIFENBERG [1], [2], and [3] concerning the higher dimensional PLATEAU problem and the author's extension of these results to varieties on a Riemannian manifold.

Notations. For the most part, we use standard notations. G and D will denote domains which are bounded unless otherwise specified. We denote the boundary of D by ∂D and its closure by $\bar{D}$. We shall often use the notation $D \subset\subset G$ to mean that $\bar{D}$ is compact and $\bar{D} \subset G$. $B(x_0, R)$ denotes the ball with center at x_0 and radius R. γ_ν and Γ_ν denote the ν-measure and $(\nu - 1)$-measure of $B(0,1)$ and $\partial B(0,1)$, respectively. We often denote $\partial B(0,1)$ by Σ. Most of the time (unless otherwise specified) we let R_q be q-dimensional number space with the usual metric and abbreviate $B(0,R)$ to B_R, denote by σ the $(\nu - 1)$-plane $x^\nu = 0$, and define

$$\begin{aligned} R_\nu^+ &= \{x \mid x^\nu > 0\}, & R_\nu^- &= \{x \mid x^\nu < 0\} \\ G_R &= B_R \cap R_\nu^+, & \Sigma_R &= \partial B_R \cap R_\nu^+, \quad \sigma_R = B_R \cap \sigma \\ G_R^- &= B_R \cap R_\nu^-, & \Sigma_R^- &= \partial B_R \cap R_\nu^-. \end{aligned} \tag{1.2.1}$$

If S is a set in R_q, $|S|$ denotes its Lebesgue q-measure; if x is a point, $d(x, S)$ denotes the distance of x from S. We define

$$[a, b] = \{x \mid a^\alpha \leq x^\alpha \leq b^\alpha, \quad \alpha = 1, \ldots, \nu, \ x \in R_\nu\}.$$

In the case of boundary integrals, we often use dx'_α to denote $n_\alpha \, dS$ where dS is the surface area and n_α is the α-th component of the *exterior normal*. We say that a function $u \in C^n(G)$ iff (if and only if) u and its partial derivatives of order $\leq n$ are continuous on G and $u \in C^n(\bar{G})$ iff $u \in C^n(G)$ and each of its derivatives of order $\leq n$ can be extended to be continuous on $\bar{G}$. If $0 < \mu \leq 1$, $u \in C_\mu^n(G)$ (or $C_\mu^n(\bar{G})$) $\Leftrightarrow$ (i.e. iff) $u \in C^n(G)$ (or $C^n(\bar{G})$) and all the derivatives of order $\leq n$ satisfy a Hölder (Lipschitz if $\mu = 1$) condition on each compact subset of G (or on the whole of $\bar{G}$ as extended). If $u \in C_\mu^0(\bar{G})$, then $h_\mu(u, \bar{G}) = \sup |x_2 - x_1|^{-\mu} \cdot |u(x_2) - u(x_1)|$ for x_1 and $x_2 \in \bar{G}$ and $x_1 \neq x_2$. A domain G is said to be of class C^n (or C_μ^n, $0 < \mu \leq 1$) iff G is bounded and each point P_0 of ∂G is in a neighborhood $\mathfrak{n}$ on $\bar{G}$ which can be mapped by a $1-1$ mapping of class C^n (or C_μ^n), together with its inverse, onto $G_R \cup \sigma_R$ for some R in such a way that P_0 corresponds to the origin and $\mathfrak{n} \cup \partial G$ corresponds to σ_R. If $u \in C^1(G)$, we denote its derivatives $\partial u/\partial x^\alpha$ by $u_{,\alpha}$. If $u \in C_2(G)$, then $\nabla^2 u$ denotes the tensor $u_{,\alpha\beta}$ where α and β run independently from to ν. Likewise $\nabla^3 u = \{u_{,\alpha\beta\gamma}\}$, etc., and $|\nabla^2 u|^2 = \sum_{\alpha,\beta} |u_{,\alpha\beta}|^2$, etc. If G is also of class C^1, then Green's theorem becomes (in our notations)

$$\int_G u_{,\alpha}(x)\, dx = \int_{\partial G} u\, n_\alpha\, dS = \int_{\partial G} u\, dx'_\alpha.$$

Sometimes when we wish to consider u as a function of some single x^α, we write $x = (x^\alpha, x'_\alpha)$ and $u(x) = u(x^\alpha, x'_\alpha)$ where x'_α denotes the remaining x^β. One dimensional or $(\nu - 1)$-dimensional integrals are then indicated as might be expected. We often let α denote a "multi-index", i.e. a vector $(\alpha_1, \ldots, \alpha_\nu)$ in which each α_i is a non-negative integer. We then define

$$|\alpha| = \alpha_1 + \cdots + \alpha_\nu,\ D^\alpha u = \frac{\partial^{|\alpha|} u}{(\partial x^1)^{\alpha_1} \dots (\partial x^\nu)^{\alpha_\nu}}\ (u \in C^{|\alpha|}(G))$$

$$\alpha! = (\alpha_1!) \dots (\alpha_\nu!),\ C_\alpha = \frac{|\alpha|!}{\alpha!},\ \xi^\alpha = (\xi^1)^{\alpha_1} \dots (\xi^\nu)^{\alpha_\nu}.$$

Using this notation

$$|\nabla^m u|^2 = \sum_{|\alpha|=m} C_\alpha |D^\alpha u|^2.$$

We shall denote constants by C or Z with or without subscripts. These constants will, perhaps depend on other constants; in this case we may write $C = C(h, \mu)$ if C depends only on h and μ, for example. However, even though we may distinguish between different constants in some discussion by inserting subscripts, there is no guarantee that C_2, for example, will always denote the same constant. We sometimes denote the support of u by spt u. We denote by $C_c^\infty(G)$, $C_c^n(G)$, and $C_{\mu c}^n(G)$ the sets of functions in $C^\infty(G)$, $C^n(G)$, or $C_\mu^n(G)$, respectively, which have support in G (i.e. which vanish on and near ∂G). But it is handy to say that u has support in $G_R \cup \sigma_R \Leftrightarrow u$ vanishes on and near Σ_R (see 1.2.1); we allow $u(x)$ to be $\neq 0$ on σ_R.

1.3. Very brief historical remarks

Problems in the calculus of variations which involve only single integrals ($\nu = 1$) have been discussed at least since the time of the BERNOULLI's. Although there was some early consideration of double integrals, it was RIEMANN who aroused great interest in them by proving many interesting results in function theory by assuming DIRICHLET's *principle* which may be stated as follows: *There is a unique function which minimizes the* DIRICHLET *integral among all functions of class C^1 on a domain G and continuous on $\bar{G}$ which takes on given values on the boundary ∂G and, moreover, that function is harmonic on G.*

RIEMANN's work was criticized on the grounds that just because the integral was bounded below among the competing functions it didn't follow that the greatest lower bound was *taken on* in the class of competing functions. In fact an example was given of a (1-dimensional) integral of the type (1.1.1) for which there is no minimizing function and another was given of continuous boundary values on the unit circle such that $D(z, G) = +\infty$ for every z as above having those boundary values.

The first example is the integral (see COURANT [3])

$$I(z, G) = \int_0^1 \left[1 + \left(\frac{dz}{dx}\right)^2\right]^{1/4} dx,\ G = (0,1), \tag{1.3.1}$$

the admissible functions z being those $\in C^1$ on $[0,1]$ with

$$z(0) = 0 \text{ and } z(1) = 1.$$

Obviously $I(z, G) > 1$ for every such z, $I(z, G)$ has no upper bound and if we define

$$z_r(x) = \begin{cases} 0 & , 0 \leq x \leq r \qquad , 0 < r < 1, \\ -1 + [1 + 3(x-r)^2/(1-r)^2]^{1/2}, & r \leq x \leq 1 \end{cases}$$

we see that $I(z_r, G) \to 1$ as $r \to 1^-$.

The second example is the following (see COURANT [3]): It is now known that Dirichlet's principle holds for a circle and that each function harmonic on the unit circle has the form

$$(1.3.2) \qquad w(r, \Theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\Theta + b_n \sin n\Theta), \ (a_n, b_n \text{ const}),$$

in polar coordinates and that the Dirichlet integral is

$$(1.3.3) \qquad D(w, G) = \pi \sum_{n=1}^{\infty} n(a_n^2 + b_n^2)$$

provided this sum converges. But if we define

$$a_n = k^{-2} \text{ if } n = k! \quad , a_n = b_n = 0 \text{ otherwise,}$$

we see that the series in (1.3.2) converges uniformly but that in (1.3.3) reduces to

$$\pi \sum_{k=1}^{\infty} \frac{k!}{k^4}$$

which diverges.

DIRICHLET's principle was established rigorously in certain important cases by HILBERT, LEBESGUE [2] and others shorly after 1900. That was the beginning of the so-called "direct methods" of the calculus of variations of which we shall say more later.

There was renewed interest in one dimensional problems with the advent of the MORSE theory of the critical points of functionals in which M. MORSE generalized his theory of critical points of functions defined on finite-dimensional manifolds [1] to certain functionals defined on infinite-dimensional spaces [2], [3]. He was able to obtain the MORSE inequalities between the numbers of possibly "unstable" (i.e. critical but not minimizing) geodesics (and unstable minmal surfaces) having various indices (see also MORSE and TOMPKINS, [1]—[4]). Except for the latter (which could be reduced to the case of curves), MORSE's theory was applied mainly to one-dimensional problems. However, within the last two years, SMALE and PALAIS and SMALE have found a modification of MORSE's theory which is applicable to a wide class of multiple integral problems.

Variational methods are beginning to be used in differential geometry. For example, the author and Eells (see MORREY and EELS, MORREY,

[11] and Chapter 7) developed the HODGE theory ([1], [2]) by variational methods (HODGE's original idea [1]). HÖRMANDER [2], KOHN [1], SPENCER (KOHN and SPENCER), and the author (MORREY [19], [20]) have applied variational techniques to the study of the $\bar{\partial}$-Neumann problem for exterior differential forms on complex analytic manifolds (see Chapter 8; the author encountered this problem in his work on the analytic embedding of real-analytic manifolds (MORREY [13]). Very recently, EELLS and SAMPSON have proved the existence of "harmonic" mappings (i.e. mappings which minimize an intrinsic Dirichlet integral) from one compact manifold into a manifold having negative curvature. Since the inf. of this integral is zero if the dimension of the compact manifold >2, they found it necessary to use a gradient line method which led to a non-linear system of parabolic equations which they then solved; the curvature restriction was essential in their work.

1.4. The Euler equations

After a number of special problems had been solved, EULER deduced in 1744 the first general necessary condition, now known as EULER's equation, which must be satisfied by a minimizing or maximizing arc. His derivation, given for the case $N = \nu = 1$, proceeds as follows: Suppose that the function z is of class C^1 on $[a, b]$ $(= G)$ minimizes (for example) the integral $I(z, G)$ among all similar functions having the same values at a and b. Then, if ζ is *any* function of class C^1 on $[a, b]$ which vanishes at a and b, the function $z + \lambda\zeta$ is, for every λ, of class C^1 on $[a, b]$ and has the same values as z at a and b. Thus, if we define

$$(1.4.1)\qquad \varphi(\lambda) = I(z + \lambda\zeta, G) = \int_a^b f[x, z(x) + \lambda\zeta(x), z'(x) + \lambda\zeta'(x)]\, dx$$

φ must take on its minimum for $\lambda = 0$. If we assume that f is of class C^1 in its arguments, we find by differentiating (1.4.1) and setting $\lambda = 0$ that

$$(1.4.2)\qquad \int_a^b \{\zeta'(x)\cdot f_p[x, z(x), z'(x)] + \zeta(x) f_z[x, z(x), z'(x)]\}\, dx = 0 \qquad \left(f_p = \frac{\partial f}{\partial p}, \text{ etc.}\right).$$

The integral in (1.4.2) is called the *first variation* of the integral I; it is supposed to vanish for every ζ of class C^1 on $[a, b]$ which vanishes at a and b. *If we now assume that f and z are of class C^2 on* $[a, b]$ (EULER had no compunctions about this) we can integrate (1.4.2) by parts to obtain

$$(1.4.3)\qquad \int_a^b \zeta(x)\cdot\left\{f_z - \frac{d}{dx} f_p\right\} dx = 0, \quad f_p = f_p[x, z(x), \nabla z(x)], \text{ etc.}$$

Since (1.4.3) holds for all ζ as above, it follows that the equation

$$\frac{d}{dx} f_p = f_z \tag{1.4.4}$$

must hold. This is *Euler's equation* for the integral I in this simple case. If we write out (1.4.4) in full, we obtain

$$f_{pp} \cdot z'' + f_{pz} z' + f_{px} = f_z \tag{1.4.5}$$

which shows that Euler's equation is non-linear and of the second order. It is, however, linear in z''; equations which are linear in the derivatives of highest order are frequently called ***quasi-linear***. The equation evidently becomes singular whenever $f_{pp} = 0$. Hence ***regular*** variational problems are those for which f_{pp} never vanishes; in that case, it is assumed that $f_{pp} > 0$ which turns out to make minimum problems more natural than maximum problems.

It is clear that this derivation generalizes to the most general integral (1.1.1) provided that f and the minimizing (or maximizing, etc.) function z is of class C^2 on the closed domain G which has a sufficiently smooth boundary. Then, if z minimizes I among all (vector) functions of class C^1 with the same boundary values and ζ is any such vector which vanishes on the boundary or G, it follows that $z + \lambda\zeta$ is a "competing" or "admissible" function for each λ so that if φ is defined by

$$\varphi(\lambda) = I(z + \lambda\zeta, G) \tag{1.4.6}$$

then $\varphi'(0) = 0$. This leads to the condition that

$$\int_G \sum_{i=1}^{N} \left\{ \sum_{\alpha=1}^{\nu} \zeta^i_{,\alpha} f_{p^i_\alpha} + \zeta^i f_{z^i} \right\} dx = 0 \tag{1.4.7}$$

for all ζ as indicated. The integral in (1.4.7) is the *first variation* of the general integral (1.1.1). Integrating (1.4.7) by parts leads to

$$\int_G \sum_{i=1}^{N} \zeta^i \cdot \left\{ f_{z^i} - \sum_{\alpha=1}^{\nu} \frac{\partial}{\partial x^\alpha} f_{p^i} \right\} dx = 0.$$

Since this is zero for all vectors ζ, it follows that

$$\sum_{\alpha=1}^{\nu} \frac{\partial}{\partial x^\alpha} f_{p^i_\alpha} = f_{z^i}, \quad i = 1, \ldots, N \tag{1.4.8}$$

which is a quasi-linear system of partial differential equations of the second order. In the case $N = 1$, it reduces to

$$\sum_{\alpha=1}^{\nu} \frac{\partial}{\partial x^\alpha} f_{p_\alpha} = f_z, \quad \text{or}$$

$$\sum_{\alpha,\beta=1}^{\nu} f_{p_\alpha p_\beta} z_{,\alpha\beta} + \sum_{\alpha=1}^{\nu} (f_{p_\alpha z} z_{,\alpha} + f_{p_\alpha x^\alpha}) = f_z. \tag{1.4.9}$$

The equation (1.4.9) is evidently singular whenever the quadratic form

(1.4.10) $$\sum_{\alpha,\beta} f_{p_\alpha p_\beta}(x, z, p)\,\lambda_\alpha \lambda_\beta\,*$$

in λ is degenerate.

We notice from (1.4.5) that if $N = \nu = 1$ and f depends only on p and the problem is regular, then Euler's equation reduces to

$$z'' = 0.$$

In general, if f depends only on $p(= p^i_\alpha)$, Euler's equation has the form

$$\sum_{j,\alpha,\beta} f_{p^i_\alpha p^j_\beta} z^j_{,\alpha\beta} = 0, \quad i = 1, \ldots, N$$

and every linear vector function is a solution. In particular, if $N = 1$ and $f = |p|^2$, Euler's equation is just Laplaces equation

$$\Delta z \equiv \sum_\alpha z_{,\alpha\alpha} = 0.$$

In case $f = (1 + |p|^2)^{1/4}$ as in the first example in § 1.3, we see that

$$4 f_{pp} = (2 - p^2)(1 + p^2)^{-7/4}$$

which is not always > 0. On the other hand $f_{pp} > 0$ if $|p| < \sqrt{2}$ so classical results which we shall discuss later (see § 1.6) show that the linear function $z(x) = x$ minimizes the integral among all arcs having $|z'(x)| \leq \sqrt{2}$.

We now revert to equation (1.4.9). If we take, for instance, $N = 1$, $\nu = 2$, $f = p_1^2 - p_2^2$, then (1.4.9) becomes

$$z_{,11} - z_{,22}\left(\equiv \frac{\partial^2 z}{\partial (x^1)^2} - \frac{\partial^2 z}{\partial (x^2)^2}\right) = 0$$

which is of *hyperbolic type*. Moreover, the integral (1.1.1) with this f obviously has no minimum or maximum, whatever boundary values are given for z. Anyhow, it is well known that boundary value problems are not natural for equations of hyperbolic type. If $\nu > 2$ a greater variety of types may occur, depending on the signature of the quadratic form (1.4.10). A similar objection occurs in all cases except those in which the form (1.4.10) is *positive definite* or negative definite; we shall restrict ourselves to the case where it is positive definite. In this case Euler's equation is of *elliptic type*. The choice of this condition on f is re-enforced by analogy with the case $\nu = 1$; in that case $f_{pp} \geq 0$ *implies the convexity* (see § 1.8) *of f as a function of p* for each (x, z) and the non-negative definiteness of the form (1.4.10) is equivalent to the convexity of f as a function of $p_1, \ldots, p_\nu$ for each set $(x^1, \ldots, x^\nu, z)$. Our choice is re-enforced further by the classical derivation given in the next section.

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* Greek indices are summed from 1 to ν and Latin indices are summed from 1 to N. Hereafter we shall usually employ the summation convention in which repeated indices are summed and summation signs omitted.