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A collection of informal reports and seminars
Edited by A. Dold, Heidelberg and B. Eckmann, Zürich

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Symposium on Semantics of Algorithmic Languages

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Symposium on Semantics of Algorithmic Languages

Edited by E. Engeler, University of Minnesota

Springer-Verlag
Berlin · Heidelberg · New York 1971

AMS Subject Classifications (1970): 02E10, 02F43, 68A05, 68A10, 68A30

ISBN 3-540-05377-8 Springer-Verlag Berlin · Heidelberg · New York
ISBN 0-387-05377-8 Springer-Verlag New York · Heidelberg · Berlin

Offsetdruck: Julius Beltz, Weinheim/Bergstr.

PREFACE

During the last few years, a number of interesting results and promising ideas have been generated in the area of semantical aspects of programming languages. We felt that we would do a real service to this emerging field by calling a write-in symposium and collecting a number of representative contributions covering the various aspects.

We are happy to present here the results of this endeavor in the form of Lecture Notes. It is understood that many of the contributions represent work in progress and that some may appear elsewhere in final form. We take this opportunity to thank the contributors and Springer Verlag for their exceptionally prompt collaboration.

Minneapolis, October 1970

Erwin Engeler

P.S. The papers are arranged alphabetically according to authors, a joint bibliography is located at the end of the volume.

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AXIOM SYSTEMS FOR SIMPLE ASSIGNMENT STATEMENTS*

by

J. W. de Bakker

1. Introduction

In this paper we present a number of axiom systems for simple assignment statements and investigate some of their properties and mutual relations.

Simple assignment statements are statements of the form $a := b$, $x := y$, etc. We give a formal definition (in section 2) of the effect of a sequence of simple assignment statements upon a variable. Two such sequences are called equivalent if they have the same effect upon all variables. In section 3, we single out four equivalences as axioms, and give a number of rules of inference which allow us to derive other equivalences. We then show that the axiom system is complete in the sense that if two sequences have the same effect upon all variables, then their equivalence is derivable in the system, and vice versa. The axioms are also shown to be independent. In section 5, we investigate the possibility of replacing the four axioms by a smaller number. First we show that three axioms suffice, and then we introduce an infinity of pairs of axioms, all equipollent with the original four. Some variations of these systems are discussed in section 6. In section 7, we give a one-axiom system, but with an extension of the main rule of inference.

Axiomatic characterizations of programming concepts in terms of equivalences have been given by McCarthy (1962, 1963a) as extended by Kaplan (1968b) , and Igarashi (1964, 1968). Our paper is closest to Igarashi's, where (general) assignment, conditionals, and goto statements are treated. Igarashi also gives several

*This paper is a somewhat modified, and slightly extended, version of de Bakker (1968).

completeness theorems, of which ours is a special case (our proof is different, however). An axiomatic approach to programming concepts, including assignment, is also taken by Hoare (1969), but he does not take equivalence of statements as his starting point. References to various other approaches to assignment, not directly of an axiomatic nature, can be found in our survey paper (de Bakker, 1969) and in the bibliography of London (1970b) which also contains more recent material.

2. Definitions

Let V be an infinite set, V^2 the set of all ordered pairs of elements of V, and V^{2*} the set of all finite non-empty sequences of elements of V^2. The elements of V are denoted by lower case letters, possibly with indices, e.g., $a,b,\ldots,s_1,t_2,\ldots,x,y,z$; the elements of V^2 are denoted by pairs such as ab, s_1t_2, or xy, and the elements of V^{2*} are denoted by sequences such as $ab\ cd$, s_1t_2, or $xy\ yz\ zx$. $S,S_1,S_2,\ldots$ stand for arbitrary elements of V^{2*}.

Definition 2.1 1. The elements of V are called variables.

2. The elements of V^2 are called assignment statements.

3. The elements of V^{2*} are called sequences of assignment statements.

The elements of V correspond to the (simple) variables of, e.g., ALGOL 60; the elements of V^2 to assignment statements such as $a:=b$, $s_1:=t_2$, or $x:=y$; and the elements of V^{2*} to sequences of assignment statements such as $a:=b;c:=d$, or $s_1:=t_2$, or $x:=y;y:=z;z:=x$.

Definition 2.2 Let $S\in V^2$. $p_i(S)$, $i=1,2$, is the ith element of the ordered pair S.

Definition 2.3 Let $S\in V^{2*}$. The set of left parts, $\lambda(S)$, and the set of right parts $\rho(S)$, are defined by:

1. If $S\in V^2$, then $\lambda(S)=\{p_1(S)\}$ and $\rho(S)=\{p_2(S)\}$.

2. If $S=S_1S_2$, $S_1\in V^2$, $S_2\in V^{2*}$, then

$$\lambda(S)=\lambda(S_1)\cup\lambda(S_2) ,$$

and

$$\rho(S)=\rho(S_1)\cup\rho(S_2) .$$

Definition 2.4 Let $S \in V^{2*}$. The length $\ell(S)$ of S is defined by:

1. If $S \in V^2$, then $\ell(S) = 1$.
2. If $S = S_1 S_2$, $S_1 \in V^2$, $S_2 \in V^{2*}$, then $\ell(S) = 1 + \ell(S_2)$.

For the simple assignment statements we are concerned with in our paper, it is easy to give a formal description of their usual meaning, by defining the effect E of a sequence S upon a variable a .

Definition 2.5 The function $E: V \times V^{2*} \to V$ is defined by:

1. Let $a \in V$ and $S \in V^2$. Then

$$\begin{aligned} E(a,S) &= p_2(S) \ , \text{ if } a = p_1(S) \ , \\ &= a \ , \text{ if } a \neq p_1(S) \ . \end{aligned}$$

2. Let $a \in V$ and $S = S_1 S_2$, with $S_1 \in V^{2*}$ and $S_2 \in V^2$. Then

$$E(a,S) = E(E(a,S_2),S_1) \ .$$

Examples: $E(a , ab) = b$, $E(a , bc) = a(b \neq c)$, and $E(c , ab\ bc\ ca) = E(E(c , ca) , ab\ bc)$ $= E(a , ab\ bc) = E(E(a , bc) , ab) = E(a , ab) = b$.

Next, the notion of equivalence of two sequences of assignment statements is defined in terms of the function E .

Definition 2.6 Let $S_1, S_2 \in V^{2*}$. S_1 and S_2 are called equivalent if for all $a \in V$, $E(a,S_1) = E(a,S_2)$.

Example 1: ab bc ca ab and ac cb ba are equivalent, since

$$E(a , ab\ bc\ ca\ ab) = c = E(a , ac\ cb\ ba) \ ,$$
$$E(b , ab\ bc\ ca\ ab) = c = E(b , ac\ cb\ ba) \ ,$$
$$E(c , ab\ bc\ ca\ ab) = b = E(c , ac\ cb\ ba) \ ,$$

and for all $d \neq a,b,c$,

$$E(d , ab\ bc\ ca\ ab) = d = E(d , ac\ cb\ ba) \ .$$

Example 2: ab bc and bc ab are not equivalent, since

$$E(a , ab\ bc) = b \ , \text{ and } E(a , bc\ ab) = c \ .$$

3. An Axiomatic Theory for Equivalence

We now introduce a formal axiomatic theory $\mathfrak{T}$ for the equivalence of simple assignment statements. Well-formed formulas of the theory are expressions of the form $S_1 = S_2$, with $S_1, S_2 \in V^{2*}$. The axioms of the system are:

A_1: For all $a, b \in V$, $ab\ ba = ab$.

A_2: For all $a, b, c \in V$ with $a \neq c$, $ab\cdot ac = ac$.

A_3: For all $a, b, c \in V$, $ab\ ca = ab\ cb$.

A_4: For all $a, b, c \in V$, $ab\ cb = cb\ ab$.

The rules of inference are:

R_1: If $S_1ac = S_2ac$, and $S_1bd = S_2bd (a \neq b)$, then $S_1 = S_2$.

R_2: If $S_1 = S_2$, then $S_2 = S_1$. If $S_1 = S_2$ and $S_2 = S_3$, then $S_1 = S_3$.

R_3: If $S_1 = S_2$, then $SS_1 = SS_2$ and $S_1S = S_2S$.

Remarks:

1. The set of axioms $\{A_1, A_2, A_3, A_4\}$ is denoted by $\mathcal{a}$.
2. Rule R_1 may be understood intuitively as follows: If S_1 and S_2 have the same effect upon all variables with the possible exception of a , if they have the same effect upon all variables with the possible exception of b , and if $a \neq b$, then S_1 and S_2 have the same effect upon all variables, i.e., $S_1 = S_2$.
3. We shall use formulae of the form $S_1 = S_2 = S_3 = \ldots$ as an abbreviation for $S_1 = S_2$ and $S_2 = S_3$ and $S_3 = \ldots$.

Lemma 3.1. $S = S$.

Proof.

(1) $ab\ ba = ab$, A_1;

(2) $ab = ab\ ba$, (1), R_2;

(3) $ab = ab$, (2), (1), R_2;

(4) $S = S$, (3), R_3.

From now on, the rules R_2 and R_3 will be used without explicit mentioning.

Lemma 3.2. $ab\ ab = ab$.

Proof. $ab\ ab = ab\ ba\ ab = ab\ ba = ab$, by A_1, A_1, A_1 .

Lemma 3.3. If $a \neq c$, $a \neq d$, and $b \neq c$, then $ab\ cd = cd\ ab$.

Proof.

(1) $ab\ cd\ cb = ab\ cb\ (b \neq c)$, A_2;

(2) $cd\ ab\ cb = cd\ cb\ ab = cb\ ab\ (b \neq c)$, A_4, A_2;

(3) $ab\ cd\ cb = cd\ ab\ cb\ (b \neq c)$, (1), (2), A_4;

(4) $ab\ cd\ ad = cd\ ab\ ad\ (a \neq d)$, similar to (3);

(5) $ab\ cd = cd\ ab\ (a \neq c, a \neq d, b \neq c)$, (3), (4), R_1.

Lemma 3.4. If $\lambda(S_1) \cap \lambda(S_2) = \lambda(S_1) \cap \rho(S_2) = \lambda(S_2) \cap \rho(S_1) = \phi$, then $S_1 S_2 = S_2 S_1$.

Proof. By repeated application of lemma 3.3.

(Using the completeness theorem of section 4, it can be proved that the assertion of the lemma also holds with "if" replaced by "only if".)

Lemma 3.5. $aa\ bc = bc\ aa = bc$.

Proof. 1. First we show that $aa\ bc = bc$.

(1) $aa\ ba = ba\ aa = ba\ ab = ba$, A_4, A_3, A_1;

(2) $aa\ ac = ac$, A_2, A_1;

(3) $aa\ bc\ ac = aa\ ac\ bc = ac\ bc = bc\ ac$, A_4, (2), A_4;

(4) $aa\ bc\ ba = aa\ ba = ba = bc\ ba\ (a \neq b)$, A_2, (1), A_2;

(5) $aa\ bc = bc\ (a \neq b)$, (3), (4), R_1;

(6) $aa\ bc = bc$, (2), (5).

2. Now we prove that $bc\ aa = bc$.

(7) $bc\ aa = aa\ bc = bc\ (a \neq b, a \neq c)$, lemma 3.3 and part 1;

(8) $ac\ aa = ac\ ac = ac$, A_3 and lemma 3.2;

(9) $ba\ aa = ba\ ab = ba$, A_3, A_1;

(10) $bc\ aa = bc$, (7), (8), (9).

Lemma 3.6. $aa\ S = S\ aa = S$.

Proof. Follows by lemma 3.5.

The next lemma is included because its proof illustrates the method used in the proof of the completeness theorem in the next section. The lemma shows the effect of two successive interchanges of the two variables b and c .

Lemma 3.7. $ab\ bc\ ca\ ab\ bc\ ca = ac\ (a \neq c)$.

Proof. It is easy to verify, using the previous lemmas, that the assertion holds if $a = b$ or $b = c$. Now suppose that a, b, c differ from each other. Let x, y, z be three variables, different from a, b, c . Then

$$\begin{array}{llr}
ab\ bc\ ca\ ab\ bc\ ca\ ax\ by & = ab\ bc\ ca\ ab\ bc\ by\ ca\ ax & = \\
ab\ bc\ ca\ ab\ by\ ca\ ax & = ab\ bc\ ca\ ab\ ca\ by\ ax & = \\
ab\ bc\ ca\ ab\ cb\ by\ ax & = ab\ bc\ ca\ cb\ ab\ by\ ax & = \\
ab\ bc\ cb\ ab\ by\ ax & = ab\ bc\ ab\ by\ ax & = \\
ab\ bc\ ac\ by\ ax & = ab\ ac\ bc\ by\ ax & = \\
ac\ by\ ax & = ac\ ax\ by &
\end{array}$$

by the axioms and lemma 3.3. Hence,

(1) $ab\ bc\ ca\ ab\ bc\ ca\ ax\ by = ac\ ax\ by$.

Similarly, it is proved that

(2) $ab\ bc\ ca\ ab\ bc\ ca\ by\ cz = ac\ by\ cz$,

and

(3) $ab\ bc\ ca\ ab\ bc\ ca\ ax\ cz = ac\ ax\ cz$.

By (1), (2), and R_1,

(4) $ab\ bc\ ca\ ab\ bc\ ca\ by = ac\ by$.

By (1), (3), and R_1,

(5) $ab\ bc\ ca\ ab\ bc\ ca\ ax = ac\ ax$.

By (4), (5), and R_1, $ab\ bc\ ca\ ab\ bc\ ca = ac$.

Remark. Lemma 3.7 is a fundamental property of assignment. In fact, it may replace axiom A_2:

(1) $ab\ ab = ab$, lemma 3.2;

(2) $ab\ ac = ab\ ab\ bc\ ca\ ab\ bc\ ca = ab\ bc\ ca\ ab\ bc\ ca = ac\ (a \neq c)$, lemma 3.7, (1), and lemma 3.7.

Hence, A_2 can be proved from A_1 and lemma 3.7.

4. Completeness and Independence of the Axiom System

Theorem 4.1. (Completeness theorem) Two sequences of assignment statements S_1 and S_2 are equivalent (in the sense of definition 2.6) if and only if $S_1 = S_2$ is a theorem of $\mathcal{J}$.

For the proof, we need an auxiliary theorem. We use the notation $\prod_{j=1}^{m} S_j$ as an abbreviation for $S_1 S_2 \ldots S_m$.

Theorem 4.2. Let $S \in V^{2*}$, $\lambda(S) = \{a_1, a_2, \ldots, a_m\}$, $m \geq 1$. Let X be a subset of V such that $X \cap \lambda(S) = \phi$. Then for each i, $1 \leq i \leq m$, and each $x_1, x_2, \ldots, x_m \in X$,

$$S \prod_{\substack{j=1 \\ j \neq i}}^{m} a_j x_j = a_i E(a_i, S) \prod_{\substack{j=1 \\ j \neq i}}^{m} a_j x_j .$$

Proof. We use induction on the length of S

1. $\ell(S) = 1$, i.e., $S = ab$, for some $a, b \in V$. Then, clearly $ab = aE(a, ab)$.

2. Let the assertion be proved for all $S' \in V^{2*}$ with $\ell(S') = n$. Let S be an element of V^{2*}, with $\ell(S) = n+1$. Then $S = S'\ ab$, for some $ab \in V^2$ and $S' \in V^{2*}$ with $\ell(S') = n$. Let $\lambda(S') = \{a_1, a_2, \ldots, a_m\}$, for some $m \leq n$. We distinguish two cases, $a \in \lambda(S')$ and $a \notin \lambda(S')$.

2.1. $a \in \lambda(S')$, i.e., $a = a_k$ for some k, $1 \leq k \leq m$. We have to prove that for each i, $1 \leq i \leq m$,

$$S'\ a_k b \prod_{\substack{j=1 \\ j \neq i}}^{m} a_j x_j = a_i\ E(a_i, S'\ a_k b) \prod_{\substack{j=1 \\ j \neq i}}^{m} a_j x_j .$$

We distinguish the cases $i = k$ and $i \neq k$.

2.1.1. $i = k$

Case (α). $b \notin \lambda(S')$. Then

$$S'\ a_k b \prod_{j \neq k} a_j x_j = S' \prod_{j \neq k} a_j x_j\ a_k b = a_k E(a_k, S') \prod_{j \neq k} a_j x_j\ a_k b =$$

$$a_k b \prod_{j \neq k} a_j x_j = a_k E(b, S') \prod_{j \neq k} a_j x_j = a_k E(a_k, S'\ a_k b) \prod_{j \neq k} a_j x_j ,$$

by repeated use of lemma 3.3, by the induction hypothesis, and since $b \notin \lambda(S')$ implies $E(b, S') = b$.

Case (β). $b = a_k$. This case follows directly from the induction hypothesis and lemma 3.6.

Case (γ). $b = a_h$, for some $h \neq k$. Let x_k be an arbitrary element of X. Then

$$S' \; a_k a_h \prod_{j\neq k} a_j x_j = S' \; a_k x_k \; a_k a_h \prod_{j\neq k} a_j x_j =$$

$$S' \prod_{j\neq h} a_j x_j \; a_k a_h \; a_h x_h = a_h \; E(a_h \, , S') \prod_{j\neq h} a_j x_j \; a_k a_h \; a_h x_h =$$

$$a_h \; E(a_h \, , S') \; a_k a_h \prod_{j\neq k} a_j x_j = a_h \; E(a_h \, , S') \, a_k \, E(a_h \, , S') a_h x_h \prod_{j\neq h,k} a_j x_j =$$

$$a_k \; E(a_h \, , S') \prod_{j\neq k} a_j x_j = a_k \; E(a_k \, , S' \; a_k a_h) \prod_{j\neq k} a_j x_j$$

by A_2, A_3, lemma 3.3, and the induction hypothesis.

2.1.2. $i \neq k$. Follows easily from the induction hypothesis.

2.2. $a \notin \lambda(S')$, i.e., $\lambda(S) = \{a_1, a_2, \ldots, a_m \, , a_{m+1}\}$, with $a = a_{m+1}$. Here we have to prove that for each i , $1 \leq i \leq m+1$,

$$S' \; a_{m+1} b \prod_{\substack{j=1 \\ j\neq i}}^{m+1} a_j x_j = a_i \; E(a_i \, , S' \; a_{m+1} b) \prod_{\substack{j=1 \\ j\neq i}}^{m+1} a_j x_j \; .$$

2.2.1. $i = m+1$. Again the cases $b \notin \lambda(S)$, $b = a_{m+1}$, and $b = a_h$, $1 \leq h \leq m$, must be distinguished. The proofs are then similar to cases (α), (β), and (γ) above.

2.2.2. $i \neq m+1$. Follows easily from the induction hypothesis. This completes the proof of theorem 4.2.

We can now give the proof of theorem 4.1.

Proof of theorem 4.1.

1. First we prove: If $S_1 = S_2$ is a theorem of $\mathfrak{J}$, then for all $a \in V$, $E(a \, , S_1) = E(a \, , S_2)$. Let $A_{\ell i}(A_{ri})$, $i = 1,2,3,4$, denote the left-hand side (right-hand side) of the axiom A_i . It is easy to verify that for all $a \in V$, $E(a \, , A_{\ell i}) = E(a \, , A_{ri})$. Moreover, it is easily established that the property of having the same effect upon all variables is preserved by application of the rules of inference.

2. Let $E(a \, , S_1) = E(a \, , S_2)$ for all $a \in V$. We prove that then $S_1 = S_2$ is a theorem of $\mathfrak{J}$. We may assume that $\lambda(S_1) = \lambda(S_2)$, say $\lambda(S_1) = \lambda(S_2) = \{a_1, a_2, \ldots, a_m\}$. (If, e.g., $a_i \in \lambda(S_1) \setminus \lambda(S_2)$, then replace S_2 by $S_2 \; a_i a_i$, etc.) Let $X \subseteq V$ be such that $X \cap \lambda(S_i) = \emptyset$, $i = 1,2$. Then, by theorem 4.1, for each i , $1 \leq i \leq m$,

$$S_1 \prod_{\substack{j=1\\ j\neq i}}^{m} a_j x_j = a_i \; E(a_i , S_1) \prod_{\substack{j=1\\ j\neq i}}^{m} a_j x_j \; ,$$

and

$$S_2 \prod_{\substack{j=1\\ j\neq i}}^{m} a_j x_j = a_i \; E(a_i , S_2) \prod_{\substack{j=1\\ j\neq i}}^{m} a_j x_j \; .$$

Since $E(a_i , S_1) = E(a_i , S_2)$, we have

$$S_1 \prod_{\substack{j=1\\ j\neq i}}^{m} a_j x_j = S_2 \prod_{\substack{j=1\\ j\neq i}}^{m} a_j x_j \; , \quad i = 1,2,\ldots,m \; .$$

Suitable repeated application of R_1 now gives $S_1 = S_2$. This completes the proof of theorem 4.1.

For the proof of the independence of our axiom system, we need a new concept and some notations. We introduce an auxiliary function F . Let N be the set of non-negative integers.

Definition 4.1. The function $F\colon V \times V^{2*} \to N$ is defined by

1. Let $a \in V$ and $S \in V^2$. Then

$$F(a,S) = 1 \; , \text{ if } \; a = p_1(S) \; \text{ and } \; a \neq p_2(S) \; ;$$
$$= 0 \; , \text{ otherwise.}$$

2. Let $a \in V$ and $S = S_1 S_2$, with $S_1 \in V^{2*}$ and $S_2 \in V^2$. Then

$$F(a , S) = F(a , S_2) + F(E(a , S_2) , S_1) \; .$$

F(a , S) may be described as the number of non-trivial steps which are made in calculating the effect of S upon a .

Example: Let a , b , c be three different variables.

$$F(b , ab\ ca\ bc\ bb) = F(b , bb) + F(E(b , bb) , ab\ ca\ bc) =$$
$$0 + F(b , ab\ ca\ bc) = F(b , bc) + F(E(b , bc) , ab\ ca) =$$
$$1 + F(c , ab\ ca) = 1 + F(c , ca) + F(E(c , ca) , ab) =$$
$$1 + 1 + F(a , ab) = 3 \; .$$

Definition 4.2. The sets of axioms $\mathfrak{a} \setminus \{A_i\}$, $i = 1,2,3,4$, are denoted by $\mathfrak{a}_i$.

In the remainder of this section and in the following sections, we shall consider sets of axioms which differ from $\mathcal{A}$. Therefore, the following notation is introduced:

Definition 4.3. Let $\mathcal{F}$ be a set of axioms, and let $S_1, S_2 \in V^{2*}$. $\mathcal{F} \vdash S_1 = S_2$ means that $S_1 = S_2$ can be derived from the set of axioms $\mathcal{F}$ by application of R_1, R_2, R_3.

Usually, it is clear from the context which set of axioms is meant. Explicit mentioning of it is then omitted. E.g., up to now, $S_1 = S_2$ always meant $\mathcal{A} \vdash S_1 = S_2$. We now prove the independence theorem.

Theorem 4.3. The set of axioms $\mathcal{A}$ is independent.

Proof. We exhibit four properties $P_i = P_i(S_1, S_2)$, $i = 1,2,3,4$, such that if $\mathcal{A}_i \vdash S_1 = S_2$, then $P_i(S_1, S_2)$ holds, but $P_i(A_{\ell i}, A_{ri})$ does not hold. ($A_{\ell i}$ and A_{ri} are the left- and right-hand sides of A_i.) These properties are:

P_1: $\lambda(S_1) = \lambda(S_2)$.

P_2: $s(S_1) = s(S_2)$, where, for $S \in V^{2*}$, $s(S)$ is the second variable of the first assignment statement in the sequence S.

P_3: For all $a \in V$, $F(a, S_1) + F(a, S_2)$ is an even number.

P_4: $f(S_1) = f(S_2)$, where, for $S \in V^{2*}$, $f(S)$ is the first variable of the first assignment statement in the sequence S.

5. Equipollent Axiom Systems

In this section, we introduce several (in fact, an infinity of) smaller sets of axioms for assignment statements, and we prove that from these systems the same equivalences can be derived as from $\mathcal{A}$. (We do not change the rules of inference, R_1, R_2, and R_3.)

Definition 5.1. Let $\mathcal{F}_1$, $\mathcal{F}_2$ be two sets of axioms for assignment statements. $\mathcal{F}_1 \Rightarrow \mathcal{F}_2$ is used as an abbreviation for: For all $S_1, S_2 \in V^{2*}$, if $\mathcal{F}_1 \vdash S_1 = S_2$, then $\mathcal{F}_2 \vdash S_1 = S_2$. $\mathcal{F}_1$ and $\mathcal{F}_2$ are called equipollent, denoted by $\mathcal{F}_1 \Leftrightarrow \mathcal{F}_2$, if $\mathcal{F}_1 \Rightarrow \mathcal{F}_2$ and $\mathcal{F}_2 \Rightarrow \mathcal{F}_1$.

In order to reduce the number of axioms, one looks for equivalences which, in some sense, combine the properties of some of the axioms A_1, A_2, A_3, and A_4. Two equivalences which combine A_3 and A_4 are

B: $ab\ ca = cb\ ab$ and B': $ab\ ca = cb\ ac$.

Combinations of A_1, A_3, and A_4 are given by

C_1: $ab\ ca\ bc = cb\ ab$ and C_1': $ab\ ca\ bc = cb\ ac$.

The structure of C_1 and C_1' suggests that one also considers

D_1: $ab\ ca\ bc\ ab = cb\ ab$ and D_1': $ab\ ca\ bc\ ab = cb\ ac$,

E_1: $ab\ ca\ bc\ ab\ ca = cb\ ab$ and E_1': $ab\ ca\ bc\ ab\ ca = cb\ ac$,

C_2: $(ab\ ca\ bc)^2 = cb\ ab$ and C_2': $(ab\ ca\ bc)^2 = cb\ ac$,

etc. ($(S)^n$ is a sequence of n times S .) The general form of these equivalences is:

C_n: $(ab\ ca\ bc)^n = cb\ ab$ and C_n': $(ab\ ca\ bc)^n = cb\ ac$,

D_n: $(ab\ ca\ bc)^n ab = cb\ ab$ and D_n': $(ab\ ca\ bc)^n ab = cb\ ac$,

E_n: $(ab\ ca\ bc)^n ab\ ca = cb\ ab$ and E_n': $(ab\ ca\ bc)^n ab\ ca = cb\ ac$.

It is easily verified that all these equivalences are indeed provable from $\mathcal{A}$.

Let $\mathcal{B} = \{A_1, A_2, B\}$ and $\mathcal{B}' = \{A_1, A_2, B'\}$,

$\mathcal{C}_n = \{A_2, C_n\}$ and $\mathcal{C}_n' = \{A_2, C_n'\}$,

$\mathcal{D}_n = \{A_2, D_n\}$ and $\mathcal{D}_n' = \{A_2, D_n'\}$,

$\mathcal{E}_n = \{A_2, E_n\}$ and $\mathcal{E}_n' = \{A_2, E_n'\}$.

We shall prove that $\mathcal{B}$, $\mathcal{B}'$, and, for each $n \geq 1$, $\mathcal{C}_n$, $\mathcal{D}_n'$, $\mathcal{E}_n$, $\mathcal{E}_n'$ are all equipollent with $\mathcal{A}$. As we shall see in section 6, this does not hold, in general, for $\mathcal{C}_n'$ and $\mathcal{D}_n$.

<u>Theorem 5.1</u>. $\mathcal{A} \Leftrightarrow \mathcal{B}$.

<u>Proof</u>. $\mathcal{B} \Rightarrow \mathcal{A}$ is clear. In order to prove $\mathcal{A} \Rightarrow \mathcal{B}$, it is sufficient to show that $\mathcal{B} \vdash A_3$ and $\mathcal{B} \vdash A_4$:

(1) $ab\ ca = ab\ ca\ ac = cb\ ab\ ac = cb\ ac = ab\ cb\ (a \neq c)$, A_1, B, A_2, B ;

(2) $ab\ aa = ab\ ab$, B ;

(3) $ab\ ca = ab\ cb$, (1), (2).

Hence, $\mathcal{B} \vdash A_3$. From A_3 and B , A_4 follows directly.

<u>Theorem 5.2</u>. $\mathcal{A} \Leftrightarrow \mathcal{B}'$.

<u>Proof</u>. $\mathcal{B}' \Rightarrow \mathcal{A}$ is clear. Proof of $\mathcal{A} \Rightarrow \mathcal{B}'$: