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Part 4

The Semiclassical Theory of the Gas Laser

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THE SEMICLASSICAL THEORY OF THE GAS LASER

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CONTENTS

1. Introduction

In a gas laser the optically active medium consists of atoms or molecules enclosed in an optical cavity resonator. The electronic excited states of the atoms or molecules form a set of discrete, unequally spaced quantum levels. The radiation in the cavity causes transitions between any two levels when the frequency of the radiation is close to their energy difference divided by $\hbar$. The unequal spacing of the levels makes it possible to maintain resonance between the radiation in the cavity and a few energy levels only. The majority of the energy levels may then be neglected or described in some average way.

The earliest gas lasers contained neon (Ne) atoms with helium (He) added to obtain more efficient excitation of the atoms. This laser type provided a large amount of insight into the experimental and theoretical problems to be encountered in laser physics. The He–Ne laser is still very important, but it is capable of only limited output power, usually less than one watt in continuous operation. Other laser types, like the argon ion laser (Ar^+) or the carbon dioxide molecular laser (CO_2), give much higher output powers; the latter has been reported to give up to 8000 watts in continuous operation. In addition to these, many other gases have been used in lasers, and the number is rapidly increasing. In many gases it is possible to use several different transitions for laser operation. Over 1000 electronic transitions in gases have now been reported to sustain laser oscillations. Most of these are, however, rather difficult to maintain and the future development is likely to centre on a few types. For a review of the situation see Bennett's article.[10]

Considering the importance of the gas laser we find it of interest to summarize our theoretical understanding of its operation. It is the purpose of the present paper to give such a summary and present the main results in sufficient detail to enable the reader to study the current literature in the field.

The electromagnetic field in the laser has a very high intensity and may consequently be treated classically. This can be justified by the correspondence principle, which states that quantum mechanics goes over into classical mechanics for large quantum numbers. The field in an optical cavity is characterized by a discrete set of eigenmodes. The atoms or molecules in the cavity are excited by a pumping mechanism involving discharge currents and atomic collisions. The actual processes taking place are very complicated and we do not consider them in any detail. The pumping causes electronic transitions to a large number of excited states, of which some pairs are in resonance with the cavity eigenmodes. These induce electronic transitions which will increase the field if atoms occur more frequently in the upper states than in the lower ones. In thermal equilibrium the lower states are always more frequently occupied, i.e. the lower energy levels have a higher population than the higher levels. Therefore the reversed situation in a laser is called population inversion. When the amplification by the resonant transitions in the gas exactly compensates the losses in the cavity, the laser oscillates in a steady state. If the gain is higher than the losses the intensity increases until a new equilibrium is achieved. Thus it is important to consider the nonlinear properties of the laser, as the saturation behaviour determines the output intensity.

In a gas laser the atoms move with a velocity distribution determined by the temperature of the gas. The atomic transition frequencies are Doppler shifted so that the gas undergoes resonance transitions over a broader range of frequencies than a stationary atom would. The resonance frequencies are distributed inhomogeneously and the transition is said to be inhomogeneously broadened.

The classical field may be determined from Maxwell's equations once the polarization of the medium is given. This polarization is, however, caused by the electromagnetic field present in the cavity and can be calculated from the Schrödinger equation. The polarization to be inserted into Maxwell's equations is thus a function of the electromagnetic field in the cavity. This dependence gives a condition for self-consistency of the field, and it determines the operating point of the laser. This model of a gas laser has been discussed in great detail by Lamb[56, 57, 58] in a third order perturbation approximation. The theory has been successful in explaining the experimental results and it

has provided useful concepts for more general treatments. We will later quote both theoretical and experimental work which show the usefulness of Lamb's approach. Another approach is used by Haken[38, 39, 40] to obtain similar results. Work has also been performed in the Soviet Union[65] and some of the earlier results are summarized by Fain and Khanin.[25]

2. Foundations of Semiclassical Laser Theory

2.1. *The electromagnetic field*

The electromagnetic field in a cavity is determined from Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{2}$$

together with

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \quad \mathbf{B} = \mu_0 \mathbf{H} \tag{3}$$

where $\mathbf{P}$ is the macroscopic polarization of the medium. The losses due to imperfect reflections in the cavity end mirrors and losses elsewhere serve to determine the intensity level of the laser and they are described by a phenomenological conductivity σ chosen to give the correct damping. We have

$$\mathbf{J} = \sigma \mathbf{E}. \tag{4}$$

Eliminating the magnetic field $\mathbf{H}$ and the current $\mathbf{J}$ in favour of the field $\mathbf{E}$ we find

$$\nabla \times (\nabla \times \mathbf{E}) + \mu_0 \sigma \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}. \tag{5}$$

The polarization $\mathbf{P}$ contains parts which oscillate with frequencies within a narrow range only, so

$$\frac{\partial^2 \mathbf{P}}{\partial t^2} \approx -\nu^2 \mathbf{P}. \tag{6}$$

where ν is the mean frequency.

The electric field is expanded in the cavity eigenfunctions as

$$\mathbf{E}(\mathbf{r}, t) = \sum_n A_n(t)\mathbf{U}_n(\mathbf{r}) \tag{7}$$

where the functions $\mathbf{U}_n(\mathbf{r})$ satisfy

$$\nabla \times [\nabla \times \mathbf{U}_n(\mathbf{r})] - \mu_0\varepsilon_0\Omega_n^2\mathbf{U}_n(\mathbf{r}) = 0 \tag{8}$$

and the boundary conditions of the cavity. The eigenfrequency of the cavity mode is Ω_n and as $\mu_0\varepsilon_0 = c^{-2}$ the corresponding wave vector is

$$|\mathbf{K}_n| = \frac{\Omega_n}{c} = \Omega_n\sqrt{\varepsilon_0\mu_0}. \tag{9}$$

The polarization in the cavity may also be expanded in the cavity eigenfunctions as

$$\mathbf{P}(\mathbf{r}, t) = \sum_n P_n(t)\mathbf{U}_n(\mathbf{r}). \tag{10}$$

Because the cavity eigenmodes form an orthogonal set it follows from (10) that

$$P_n(t) = \frac{\int \mathrm{d}^3\mathbf{r}\mathbf{P}(\mathbf{r}, t) \cdot \mathbf{U}_n(\mathbf{r})}{\int \mathrm{d}^3\mathbf{r}[\mathbf{U}_n(\mathbf{r})]^2}. \tag{11}$$

Inserting (10) and (7) into (5) and using (8) we obtain

$$\frac{\mathrm{d}^2A_n(t)}{\mathrm{d}t^2} + \frac{\sigma}{\varepsilon_0}\frac{\mathrm{d}A_n(t)}{\mathrm{d}t} + \Omega_n^2A_n(t) = \frac{\nu^2}{\varepsilon_0}P_n(t). \tag{12}$$

For the simple time dependence $A_n \exp(-i\nu t)$ we find on the left-hand side of (12) the factor

$$\begin{aligned} -\nu^2 + i\nu\frac{\sigma}{\varepsilon_0} + \Omega_n^2 &= (\Omega_n + \nu)(\Omega_n - \nu) + i\nu\frac{\sigma}{\varepsilon_0} \\ &\approx 2\nu\left(\Omega_n - \nu + i\frac{\sigma}{2\varepsilon_0}\right) \end{aligned} \tag{13}$$

because resonance interaction can take place only for $\nu \approx \Omega_n$. The half-width of the resonance is $\Delta\nu = \sigma/\varepsilon_0$ and the cavity Q value may be introduced in the usual way

$$Q = \frac{\nu}{\Delta\nu} = \frac{\nu\varepsilon_0}{\sigma}. \tag{14}$$

This relation is used to eliminate σ from (12) and gives

$$\frac{d^2A_n(t)}{dt^2} + \frac{v}{Q_n}\frac{dA_n(t)}{dt} + \Omega_n^2 A_n(t) = \frac{v^2}{\varepsilon_0} P_n(t), \tag{15}$$

where we have allowed Q_n to depend on the mode index n. In real laser cavities Q_n is very high ($\approx 10^7$).

The time dependence of A_n is at the frequency ν_n and we write

$$A_n(t) = E_n(t)\cos[\nu_n t + \varphi_n(t)] \tag{16}$$

where $E_n(t)$ and $\varphi_n(t)$ are slowly varying compared to $\cos \nu_n t$ and $\sin \nu_n t$. Only the part of P_n which oscillates at ν_n (near ν) will be important as the distance Δ between the cavity eigenfrequencies is much larger than the width of the cavity resonances, $\Delta\nu \ll \Delta$. The polarization will, however, have a phase shift with respect to the field (16) and we write

$$P_n(t) = C_n(t)\cos[\nu_n t + \varphi_n(t)] + S_n(t)\sin[\nu_n t + \varphi_n(t)] \tag{17}$$

where $C_n(t)$ and $S_n(t)$ are slowly varying. We introduce the "ansatz" (16) and (17) into (15) assuming that the small quantities $\ddot{E}, \ddot{\varphi}, \dot{\varphi}\dot{E}, Q_n^{-1}\dot{E}$ and $Q_n^{-1}\dot{\varphi}E$ can be neglected (the last two are small because Q_n is assumed large). The frequency of oscillation ν_n is very close to the cavity eigenfrequency Ω_n and we proceed as in eq. (13)

$$\nu_n + \Omega_n + \dot{\varphi} \approx 2\nu_n. \tag{18}$$

We then equate the coefficients of the rapidly varying sine and cosine terms separately and obtain

$$[\nu_n + \dot{\varphi}_n(t) - \Omega_n]E_n(t) = -\frac{1}{2}\frac{v}{\varepsilon_0} C_n(t) \tag{19}$$

$$\dot{E}_n(t) + \frac{1}{2}\frac{v}{Q_n} E_n(t) = -\frac{1}{2}\frac{v}{\varepsilon_0} S_n(t). \tag{20}$$

These two conditions serve to determine the amplitude E_n and the frequency ν_n when P_n is known. Both unknowns occur in both equations, making a simultaneous solution necessary. In practice, however, it is usually sufficient to determine the amplitude from (20) assuming that $\nu_n = \Omega_n$ and then determine $(\nu_n - \Omega_n)$ from (19).

In a laser the optical wavelength λ ($\approx 1\ \mu$) is much shorter than the length of the laser L (≈ 1 m), and the mode number n is high. The detailed structure of the cavity modes has been discussed by several authors: Fox and Li[31,32] and others [17,18,98,6,55]. The relevant features for laser operation are retained in a model where the laser cavity is assumed to be one-dimensional with length L. A coordinate system is introduced with the laser axis along the z-axis and the eigenmodes are

$$U_n(z) = \sin K_n z \tag{21}$$

with

$$K_n = \frac{\Omega_n}{c} = \frac{\pi}{L} n. \tag{22}$$

The electromagnetic field is transverse and thus we have two modes for each K_n. It is, however, often convenient to assume that we suppress one mode by Brewster windows and consider only one mode for each K_n.

2.2. *The polarization of the atoms*

The uneven spacing between the atomic† energy levels taking part in the laser action enables us to fix our attention on one pair $|a\rangle$, $|b\rangle$ as only these will be in near resonance with the cavity modes at which oscillation takes place. The pumping mechanism is considered phenomenologically by an excitation probability λ_α per unit time and unit volume into the state $|\alpha\rangle$ ($\alpha = a$ or b). The two states are assumed to decay to lower levels with decay constants γ_a and γ_b respectively; these may be calculated from a Wigner–Weisskopf approximation (see Scully and Lamb[80]). The energy difference $\hbar\omega = \hbar(W_a - W_b)$ is in near resonance with the cavity modes Ω_n which are going to perform laser oscillations. The situation is illustrated in Fig. 1.

When an atom is oscillating between the two levels $|a\rangle$, $|b\rangle$ its most general state is

$$|\psi(t)\rangle = a(t)|a\rangle + b(t)|b\rangle \tag{23}$$

† For simplicity we take the active medium to consist of atoms.

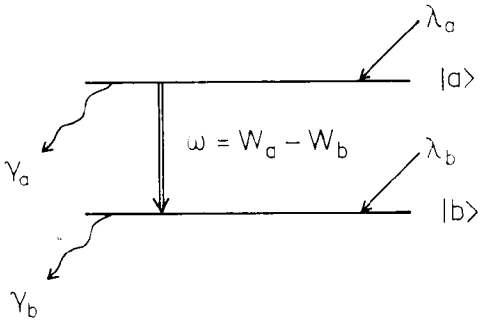

FIG. 1. The transition between two states in the atom with energies $\hbar W_a$ and $\hbar W_b$ is nearly in resonance with the cavity mode. The populations are increased by the phenomenologically described pumping rates λ_a, λ_b and decreased by decay to lower states with rates γ_a, γ_b.

where $a(t)$, $b(t)$ are complex amplitudes. As we want to consider ensembles of atoms we introduce the density matrix

$$\rho = \begin{bmatrix} a^*a & ab^* \\ ba^* & b^*b \end{bmatrix} \tag{24}$$

ρ is a reduced density matrix as we have excluded all atomic levels but $|a\rangle$, $|b\rangle$; this fact is seen in the decay of $\mathrm{Tr}\,\rho = |a|^2 + |b|^2$, due to transitions to lower states. The unperturbed Hamiltonian is

$$\hbar H_0 = \hbar \begin{bmatrix} W_a & 0 \\ 0 & W_b \end{bmatrix} \tag{25}$$

and the transitions are (in the dipole approximation) caused by

$$\sum_i \hbar V_i = -e \sum_i \mathbf{r}_i \cdot \mathbf{E} \tag{26}$$

where i runs over all atoms present in the cavity. In quantum electronics the interaction (26) is more useful than the corresponding form with the vector potential of the field. In the dipole approximation a canonical transformation shows the equivalence of (26) and the vector potential form of the interaction[66]. When we consider only one atom at a time,

we can neglect the summation in (26) and write the matrix representation

$$V = \begin{bmatrix} V_{aa} & V_{ab} \\ V_{ba} & V_{bb} \end{bmatrix}. \tag{27}$$

The transition $|a\rangle \rightarrow |b\rangle$ is a dipole transition induced by the matrix elements

$$\hbar V_{ab} = \hbar V_{ba}^{*} = -e\langle a|r|b\rangle E.$$

We assume that the matrix element $\wp = e\langle a|r|b\rangle$ is real. It follows from symmetry arguments that the diagonal elements vanish

$$\langle a|r|a\rangle = \langle b|r|b\rangle = 0$$

and the perturbation is written

$$V = -\wp E \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{28}$$

The decay parameters are introduced as the matrix

$$\Gamma = \begin{bmatrix} \gamma_a & 0 \\ 0 & \gamma_b \end{bmatrix} \tag{29}$$

and the equation of motion is

$$i\frac{\mathrm{d}}{\mathrm{d}t}\rho = [H_0 + V, \rho]_- - \tfrac{1}{2}i(\Gamma\rho + \rho\Gamma). \tag{30}$$

The off-diagonal elements are found to have a decay constant $\gamma_{ab} = \frac{1}{2}(\gamma_a + \gamma_b)$. The induced dipole moment is given by

$$\mathrm{Tr}\,(er\rho) = \mathrm{Tr}\begin{bmatrix} \wp & 0 \\ 0 & \wp \end{bmatrix}\begin{bmatrix} \rho_{aa} & \rho_{ab} \\ \rho_{ba} & \rho_{bb} \end{bmatrix} = \wp(\rho_{ab} + \rho_{ba}). \tag{31}$$

So far we have considered the density matrix for one atom only. The macroscopic polarization (10) is composed of contributions from all atoms which at the time t happen to be at the point z independently of their previous history. Let $\lambda_x(v, z_0, t_0)$ be the creation

probability of atoms in the state $|\alpha\rangle$ at the point z_0 and the time t_0 with velocity v along the z-axis. The corresponding density matrix at time $t > t_0$ is $\rho(\alpha, z_0, t_0, t)$ with

$$\rho(\alpha, z_0, t_0, t) = \begin{bmatrix} \delta_{a\alpha} & 0 \\ 0 & \delta_{b\alpha} \end{bmatrix}. \tag{32}$$

The density matrix at an arbitrary time $\hat{t}$ is then

$$\rho(v, z_0, \hat{t}) = \sum_{\alpha} \int_{-\infty}^{\hat{t}} \lambda_{\alpha}(v, z_0, t_0)\, \rho(\alpha, z_0, t_0, \hat{t})\, \mathrm{d}t_0, \tag{33}$$

where we have summed over all initial times $t_0 < \hat{t}$ with the proper probability weight $\lambda_{\alpha}(v, z_0, t_0)$ and considered atoms created in both states. The atom starting at z_0 at time t_0 with velocity v is at $\hat{t}$ at the position

$$\hat{z} = z_0 + v(\hat{t} - t_0). \tag{34}$$

We want to collect all atoms which were created at such z_0 that they at a later time $t > \hat{t}$ happen to be at the position z. We do this by summing

$$\begin{aligned} \rho(v, z, t, \hat{t}) &= \int_0^L \delta[z - \hat{z}(\hat{t}) - v(t - \hat{t})]\, \rho(v, z_0, \hat{t})\, \mathrm{d}z_0 \\ &= \sum_{\alpha} \int_0^L \mathrm{d}z_0 \int_{-\infty}^{\hat{t}} \mathrm{d}t_0\, \lambda_{\alpha}(v, z_0, t_0)\, \delta[z - z_0 - v(t - t_0)]\, \rho(\alpha, z_0, t_0, \hat{t}). \end{aligned} \tag{35}$$

We take the derivative $(\partial/\partial\hat{t})$ and obtain

$$\begin{aligned} \frac{\partial \rho(v, z, t, \hat{t})}{\partial \hat{t}} &= \sum_{\alpha} \lambda_{\alpha}[v, z - v(t - \hat{t}\,), \hat{t}\,]\, \rho[\alpha, z - v(t - \hat{t}), \hat{t}, \hat{t}] \\ &+ \sum_{\alpha} \int_0^L \mathrm{d}z_0 \int_{-\infty}^{\hat{t}} \mathrm{d}t_0\, \lambda_{\alpha}(v, z_0, t_0)\, \delta[z - z_0 - v(t - t_0)] \\ &\qquad \times \frac{\partial}{\partial \hat{t}}\, \rho(\alpha, z_0, t_0, \hat{t}). \end{aligned} \tag{36}$$

The first term is obtained from eq. (32) and the time derivative in the

second term is taken from eq. (30). The excitation probability $\lambda_a(v, z, t)$ is assumed to vary only slowly in space and time and is replaced by its average $\lambda_a(v)$. According to the expansion (7) and (16) the nth component of the electromagnetic field is

$$E_n(\mathbf{r}, \hat{t}) = A_n(\hat{t})U_n(\mathbf{r}) = E_n \sin K_n z \cos(\nu_n \hat{t} + \varphi_n) \tag{37}$$

and the perturbation is

$$\begin{aligned} V_{ab}(t) &= -(\wp/\hbar)\sum_n E_n \sin K_n z \cos(\nu_n \hat{t} + \varphi_n) \\ &= -(\wp/2\hbar)\sum_n E_n \sin K_n z \exp[i(\nu_n \hat{t} + \varphi_n)] + \text{c.c.} \end{aligned}$$

The equations of motion are then

$$\begin{aligned} &(\partial/\partial\hat{t})\,\rho_{aa}(v, z, t, \hat{t}) = \lambda_a(v) - \gamma_a \rho_{aa}(v, z, t, \hat{t}) \\ &\quad -(i\wp/\hbar)\sum_n E_n \sin K_n z(\hat{t}) \cos(\nu_n \hat{t} + \varphi_n)[\rho_{ab}(v, z, t, \hat{t}) - \rho_{ba}(v, z, t, \hat{t})] \end{aligned} \tag{38}$$

$$\begin{aligned} &(\partial/\partial\hat{t})\,\rho_{bb}(v, z, t, \hat{t}) = \lambda_b(v) - \gamma_b \rho_{bb}(v, z, t, \hat{t}) + (i\wp/\hbar) \\ &\quad \times \sum_n E_n \sin K_n z(\hat{t}) \cos(\nu_n \hat{t} + \varphi_n)[\rho_{ab}(v, z, t, \hat{t}) - \rho_{ba}(v, z, t, \hat{t})] \end{aligned} \tag{39}$$

$$\begin{aligned} &(\partial/\partial\hat{t})\,\rho_{ab}(v, z, t, \hat{t}) = -\gamma_{ab}\rho_{ab}(v, z, t, \hat{t}) - i\omega\rho_{ab}(v, z, t, \hat{t}) \\ &\quad -(i\wp/\hbar)\sum_n E_n \sin K_n z(\hat{t}) \cos(\nu_n \hat{t} + \varphi_n)[\rho_{aa}(v, z, t, \hat{t}) - \rho_{bb}(v, z, t, \hat{t})] \end{aligned} \tag{40}$$

where

$$z(\hat{t}) = z - v(t - \hat{t}) \tag{41}$$

is the position at time $\hat{t}$ of an atom which at t is at the position z. The collisions between atoms are neglected for the moment, and the atomic velocity v remains constant during the lifetime of an atom.

The subsequent calculations all rest on the equations (38)–(40). The present derivation has been developed by Lamb[56], Sargent, Lamb and Fork[74] and Stenholm and Lamb[89].

2.3. *Theory of a laser with stationary atoms*

In order to illustrate the use of eqs. (19), (20) and (38)–(40) we consider the situation where atomic motion is neglected, i.e. we consider a set of atoms with velocities very near zero. We set $v = 0$ in (38)–(40) and

find that the $\hat{t}$-dependence of the coefficients is eliminated. This, of course, greatly facilitates the solution of the equations.

We start by considering oscillations in one mode only and omit the index n. We may set the phase function $\varphi(\hat{t}) = 0$ in a steady state as it lacks physical significance for one mode. The resulting equations admit an exact solution, but for notational simplicity we assume in the following that

$$\gamma_a = \gamma_b = \gamma_{ab} = \gamma$$

implying that the two atomic levels decay equally rapidly. This is a rather unlikely coincidence, but it does not introduce any significant physical consequences because γ_a and γ_b are of the same order of magnitude in real lasers. The actual pumping rates are, however, to be replaced by effective rates[87]. This is seen when we set the electromagnetic field $E = 0$ in eqs. (38), (39) and find for the population difference probability the expression

$$\rho_{aa} - \rho_{bb} = \frac{\lambda_a}{\gamma_a} - \frac{\lambda_b}{\gamma_b}. \tag{42}$$

A necessary condition for the onset of laser oscillations is that this is positive, and if the gammas are replaced by an average value γ we have to replace the pumping probabilities λ_a and λ_b by effective probabilities λ_a' and λ_b' such that

$$\gamma^{-1}(\lambda_a' - \lambda_b') = (\lambda_a/\gamma_a) - (\lambda_b/\gamma_b).$$

After this we may neglect the difference between the gammas.

The function $\cos \nu t$ introduces rapid variations into the off-diagonal elements ρ_{ab}. We eliminate these by introducing

$$\rho_1 = \exp(i\nu t)\, \rho_{ab} \tag{43}$$

and set

$$\exp(\pm i\nu t) \cos \nu t = \tfrac{1}{2},$$

because the neglected terms vary rapidly and average to zero. This