Relativistic Quantum Mechanics

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Library of Congress Catalog Card Number 63-21778

05493

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Preface

The propagator approach to a relativistic quantum theory pioneered in 1949 by Feynman has provided a practical, as well as intuitively appealing, formulation of quantum electrodynamics and a fertile approach to a broad class of problems in the theory of elementary particles. The entire renormalization program, basic to the present confidence of theorists in the predictions of quantum electrodynamics, is in fact dependent on a Feynman graph analysis, as is also considerable progress in the proofs of analytic properties required to write dispersion relations. Indeed, one may go so far as to adopt the extreme view that the set of all Feynman graphs *is* the theory.

We do not advocate this view in this book nor in its companion volume, "Relativistic Quantum Fields," nor indeed do we advocate any single view to the exclusion of others. The unsatisfactory status of present-day elementary particle theory does not allow one such a luxury. In particular, we do not wish to minimize the importance of the progress achieved in formal quantum field theory nor the considerable understanding of low-energy meson-nucleon processes given by dispersion theory. However, we give first emphasis to the development of the Feynman rules, proceeding directly from a particle wave equation for the Dirac electron, integrated with hole-theory boundary conditions.

Three main convictions guiding us in this approach were the primary motivation for undertaking this book (later to become books):

1. The Feynman graphs and rules of calculation summarize quantum field theory in a form in close contact with the experimental numbers one wants to understand. Although the statement of the theory in terms of graphs may imply perturbation theory, use of graphical methods in the many-body problem shows that this formalism is flexible enough to deal with phenomena of nonperturbative character (for example, superconductivity and the hard-sphere Bose gas).

2. Some modification of the Feynman rules of calculation may well outlive the elaborate mathematical structure of local canonical quantum field theory, based as it is on such idealizations as fields defined at points in space-time. Therefore, let us develop these rules first, independently of the field theory formalism which in time may come to be viewed more as a superstructure than as a foundation.

3. Such a development, more direct and less formal—if less compelling—than a deductive field theoretic approach, should bring quantitative calculation, analysis, and understanding of Feynman graphs into the bag of tricks of a much larger community of physicists than the specialized narrow one of second quantized theorists. In particular, we have in mind our experimental colleagues and students interested in particle physics. We believe this would be a healthy development.

Our original idea of one book has grown in time to two volumes. In the first book, "Relativistic Quantum Mechanics," we develop a propagator theory of Dirac particles, photons, and Klein-Gordon mesons and perform a series of calculations designed to illustrate various useful techniques and concepts in electromagnetic, weak, and strong interactions. These include defining and implementing the renormalization program and evaluating effects of radiative corrections, such as the Lamb shift, in low-order calculations. The necessary background for this book is provided by a course in nonrelativistic quantum mechanics at the general level of Schiff's text "Quantum Mechanics."

In the second book, "Relativistic Quantum Fields," we develop canonical field theory, and after constructing closed expressions for propagators and for scattering amplitudes with the *LSZ* reduction technique, return to the Feynman graph expansion. The perturbation expansion of the scattering amplitude constructed by canonical field theory is shown to be identical with the Feynman rules in the first book. With further graph analysis we study analyticity properties of Feynman amplitudes to arbitrary orders in the coupling parameter and illustrate dispersion relation methods. Finally, we prove the finiteness of renormalized quantum electrodynamics to each order of the interaction.

Without dwelling further on what we do, we may list the major topics we omit from discussion in these books. The development of action principles and a formulation of quantum field theory from a variational approach, spearheaded largely by Schwinger, are on the whole ignored. We refer to action variations only in search of symmetries. There is no detailed discussion of the powerful developments in axiomatic field theory on the one hand and the purely *S*-matrix approach, divorced from field theory, on the other. Aside from a discussion of the Lamb shift and the hydrogen atom spectrum in the first book, the bound-state problem is ignored. Dynamical applications of the dispersion relations are explored only minimally. A formulation of a quantum field theory for massive vector mesons is not given—nor is a formulation of any quantum field theory with derivative couplings. Finally, we have not prepared a bibliography of all the significant original papers underlying many of the developments recorded in these books. Among the following recent excellent books or monographs is to be found the remedy for one or more of these deficiencies:

Schweber, S.: "An Introduction to Relativistic Quantum Field Theory," New York, Harper & Row, Publishers, Inc., 1961.

Jauch, J. M., and F. Rohrlich: "The Theory of Photons and Electrons," Cambridge, Mass., Addison-Wesley Publishing Company, Inc., 1955.

Bogoliubov, N. N., and D. V. Shirkov: "Introduction to the Theory of Quantized Fields," New York, Interscience Publishers, Inc., 1959.

Akhiezer, A., and V. B. Bereztetski: "Quantum Electrodynamics," 2d ed., New York, John Wiley & Sons, Inc., 1963.

Umezawa, H.: "Quantum Field Theory," Amsterdam, North Holland Publishing Company, 1956.

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1.1 Formulation of a Relativistic Quantum Theory

Since the principles of special relativity are generally accepted at this time, a correct quantum theory should satisfy the requirement of relativity: laws of motion valid in one inertial system must be true in all inertial systems. Stated mathematically, relativistic quantum theory must be formulated in a Lorentz covariant form.

In making the transition from nonrelativistic to relativistic quantum mechanics, we shall endeavor to retain the principles underlying the nonrelativistic theory. We review them briefly:[1]

1. For a given physical system there exists a state function Φ that summarizes all that we can know about the system. In our initial development of the relativistic one-particle theory, we usually deal directly with a coordinate realization of the state function, the wave function $\psi(q_i \cdots ,s_i \cdots ,t)$. $\psi(q,s,t)$ is a complex function of all the classical degrees of freedom, $q_1 \cdots q_n$, of the time t and of any additional degrees of freedom, such as spin s_i, which are intrinsically quantum-mechanical. The wave function has no direct physical interpretation; however, $|\psi(q_1 \cdots q_n,s_1 \cdots s_n,t)|^2 \geq 0$ is interpreted as the probability of the system having values $(q_1 \cdots s_n)$ at time t. Evidently this probability interpretation requires that the sum of positive contributions $|\psi|^2$ for all values of $q_1 \cdots s_n$ at time t be finite for all physically acceptable wave functions ψ.

2. Every physical observable is represented by a linear hermitian operator. In particular, for the canonical momentum p_i the operator correspondence in a coordinate realization is

$$p_i \rightarrow \frac{\hbar}{i} \frac{\partial}{\partial q_i}$$

3. A physical system is in an eigenstate of the operator Ω if

$$\Omega\Phi_n = \omega_n\Phi_n \tag{1.1}$$

where Φ_n is the nth eigenstate corresponding to the eigenvalue ω_n. For a hermitian operator, ω_n is real. In a coordinate realization the equation corresponding to (1.1) is

$$\Omega(q,s,t)\psi_n(q,s,t) = \omega_n\psi_n(q,s,t)$$

[1] See, for example, W. Pauli, "Handbuch der Physik," 2d ed., vol. 24, p. 1, J. Springer, Berlin, 1933. L. I. Schiff, "Quantum Mechanics," 2d ed., McGraw-Hill Book Company, Inc., New York, 1955. P. A. M. Dirac, "The Principles of Quantum Mechanics," 4th ed., Oxford University Press, London, 1958.

4. The expansion postulate states that an arbitrary wave function, or state function, for a physical system can be expanded in a complete orthonormal set of eigenfunctions ψ_n of a complete set of commuting operators (Ω_n). We write, then,

$$\psi = \sum_n a_n \psi_n$$

where the statement of orthonormality is

$$\sum_s \int (dq_1 \cdots) \psi_n^*(q_1 \cdots ,s \cdots ,t)\psi_m(q_1 \cdots ,s \cdots ,t) = \delta_{nm}$$

$|a_n|^2$ records the probability that the system is in the nth eigenstate.

5. The result of a measurement of a physical observable is any one of its eigenvalues. In particular, for a physical system described by the wave function $\psi = \Sigma a_n \psi_n$, with $\Omega\psi_n = \omega_n \psi_n$, measurement of a physical observable Ω results in the eigenvalue ω_n with a probability $|a_n|^2$. The average of many measurements of the observable Ω on identically prepared systems is given by

$$\langle \Omega \rangle_\psi \equiv \sum_s \int \psi^*(q_1 \cdots ,s \cdots ,t)\Omega\psi(q_1 \cdots ,s \cdots ,t)(dq_1 \cdots) = \sum_n |a_n|^2 \omega_n$$

6. The time development of a physical system is expressed by the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi \tag{1.2}$$

where the hamiltonian H is a linear hermitian operator. It has no explicit time dependence for a closed physical system, that is, $\partial H/\partial t = 0$, in which case its eigenvalues are the possible stationary states of the system. A superposition principle follows from the linearity of H and a statement of conservation of probability from the hermitian property of H:

$$\frac{d}{dt}\sum_s \int \psi^*\psi(dq_1 \cdots) = \frac{i}{\hbar}\sum_s \int (dq_1 \cdots)[(H\psi)^*\psi - \psi^*(H\psi)] = 0 \tag{1.3}$$

We strive to maintain these familiar six principles as underpinnings of a relativistic quantum theory.

1.2 Early Attempts

The simplest physical system is that of an isolated free particle, for which the nonrelativistic hamiltonian is

$$H = \frac{p^2}{2m} \tag{1.4}$$

The transition to quantum mechanics is achieved with the transcription

$$H \to i\hbar \frac{\partial}{\partial t} \tag{1.5}$$

$$\mathbf{p} \to \frac{\hbar}{i} \boldsymbol{\nabla}$$

which leads to the nonrelativistic Schrödinger equation

$$i\hbar \frac{\partial \psi(q,t)}{\partial t} = \frac{-\hbar^2 \nabla^2}{2m} \psi(q,t) \tag{1.6}$$

Equations (1.4) and (1.6) are noncovariant and therefore unsatisfactory. The left- and right-hand sides transform differently under Lorentz transformations. According to the theory of special relativity, the total energy E and momenta (p_x,p_y,p_z) transform as components of a contravariant four-vector

$$p^\mu = (p^0,p^1,p^2,p^3) = \left(\frac{E}{c}, p_x,p_y,p_z\right)$$

of invariant length

$$\sum_{\mu=0}^{3} p_\mu p^\mu \equiv p_\mu p^\mu = \frac{E^2}{c^2} - \mathbf{p} \cdot \mathbf{p} \equiv m^2c^2 \tag{1.7}$$

m is the rest mass of the particle and c the velocity of light in vacuo. The covariant notation used throughout this book is discussed in more detail in Appendix A. Here we only note that the operator transcription (1.5) is Lorentz covariant, since it is a correspondence between two contravariant four-vectors[1] $p^\mu \to i\hbar\, \partial/\partial x_\mu$.

Following this it is natural to take as the hamiltonian of a relativistic free particle

$$H = \sqrt{p^2c^2 + m^2c^4} \tag{1.8}$$

[1] We define $x^\mu = (ct,\mathbf{x})$ and $\nabla^\mu = \partial/\partial x_\mu$.

and to write for a relativistic quantum analogue of (1.6)

$$i\hbar \frac{\partial \psi}{\partial t} = \sqrt{-\hbar^2 c^2 \nabla^2 + m^2 c^4}\,\psi \tag{1.9}$$

Immediately we are faced with the problem of interpreting the square-root operator on the right in Eq. (1.9). If we expand it, we obtain an equation containing all powers of the derivative operator and thereby a nonlocal theory. Such theories are very difficult to handle and present an unattractive version of the Schrödinger equation in which the space and time coordinates appear in unsymmetrical form.

In the interest of mathematical simplicity (though perhaps with a lack of complete physical cogency) we remove the square-root operator in (1.9), writing

$$H^2 = p^2c^2 + m^2c^4 \tag{1.10}$$

Equivalently, iterating (1.9) and using the fact that[1] if $[A,B] = 0$, $A\psi = B\psi$ implies $A^2\psi = B^2\psi$, we have

$$-\hbar^2 \frac{\partial^2}{\partial t^2}\psi = (-\hbar^2\nabla^2c^2 + m^2c^4)\psi$$

This is recognized as the classical wave equation

$$\left[\Box + \left(\frac{mc}{\hbar}\right)^2\right]\psi = 0$$

where

$$\Box \equiv \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\mu} \tag{1.11}$$

Before looking further into (1.11), we note first that in squaring the energy relation we have introduced an extraneous negative-energy root

$$H = -\sqrt{p^2c^2 + m^2c^4}$$

In order to gain a simple equation, we have sacrificed positive definite energy and introduced the difficulty of "extra" negative-energy solutions. This difficulty is eventually surmounted (as we shall study in Chap. 5), and the negative-energy solutions prove capable of physical interpretation. In particular, they are associated with antiparticles. and the existence of antiparticles in nature lends strong experimental support for this procedure. So let us for a moment consider Eq. (1.10) and the inferred wave equation (1.11). Our first task is to construct a conserved current, since (1.11) is a second-order

[1] Throughout, we use the notation $[A,B] \equiv AB - BA$ for commutator brackets and $\{A,B\} \equiv AB + BA$ for anticommutator brackets.

wave equation and is altered from the Schrödinger form (1.2) upon which the probability interpretation in the nonrelativistic theory is based. This we do in analogy with the Schrödinger equation, taking ψ^* times (1.11), ψ times the complex conjugate equation, and subtracting:

$$\psi^*\left[\Box + \left(\frac{mc}{\hbar}\right)^2\right]\psi - \psi\left[\Box + \left(\frac{mc}{\hbar}\right)^2\right]\psi^* = 0$$

$$\nabla^\mu(\psi^*\nabla_\mu\psi - \psi\nabla_\mu\psi^*) = 0$$

or

$$\frac{\partial}{\partial t}\left[\frac{i\hbar}{2mc^2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right)\right] + \operatorname{div}\frac{\hbar}{2im}[\psi^*(\nabla\psi) - \psi(\nabla\psi^*)] = 0 \quad (1.12)$$

We would like to interpret $(i\hbar/2mc^2)\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right)$ as a probability density ρ. However, this is impossible, since it is not a positive definite expression. For this reason we follow the path of history[1] and temporarily discard Eq. (1.11) in the hope of finding an equation of first order in the time derivative which admits a straightforward probability interpretation as in the Schrödinger case. We shall return to (1.11), however. Although we shall find a first-order equation, it still proves impossible to retain a positive definite probability density for a single particle while at the same time providing a physical interpretation of the negative-energy root of (1.10). Therefore Eq. (1.11), also referred to frequently as the Klein-Gordon equation, remains an equally strong candidate for a relativistic quantum mechanics as the one which we now discuss.

1.3 The Dirac Equation

We follow the historic path taken in 1928 by Dirac[2] in seeking a relativistically covariant equation of the form (1.2) with positive definite probability density. Since such an equation is linear in the time derivative, it is natural to attempt to form a hamiltonian linear in the space derivatives as well. Such an equation might assume a form

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar c}{i}\left(\alpha_1\frac{\partial\psi}{\partial x^1} + \alpha_2\frac{\partial\psi}{\partial x^2} + \alpha_3\frac{\partial\psi}{\partial x^3}\right) + \beta mc^2\psi \equiv H\psi \quad (1.13)$$

[1] E. Schrödinger, *Ann. Physik*, **81,** 109 (1926); W. Gordon, *Z. Physik*, **40,** 117 (1926); O. Klein, *Z. Physik*, **41,** 407 (1927).

[2] P. A. M. Dirac, *Proc. Roy. Soc.* (*London*), **A117,** 610 (1928); *ibid.*, **A118,** 351 (1928); "The Principles of Quantum Mechanics," *op. cit.*