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An Introduction to Nonlinear Boundary Value Problems

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Preface

The theory of nonlinear boundary value problems is an extremely important and interesting area of research in differential equations. Due to the entirely different nature of the underlying physical processes, its study is substantially more difficult than that of initial value problems and consequently belongs to a third course in differential equations. Although this sophisticated branch of research has, in recent years, developed significantly, the available books are either more elementary in nature, for example the book by Baily, Shampine, and Waltman, or directed to a particular method of importance, such as that by Bellman and Kalaba. Hence it is felt that a book on an advanced level that exposes the reader to this fascinating field of differential equations and provides a ready access to an up-to-date state of this art is of immense value. With this as motivation, we present in our book a variety of techniques that are employed in the theory of nonlinear boundary value problems. For example, we discuss the following:

(i) methods that involve differential inequalities;
(ii) shooting and angular function techniques;
(iii) functional analytic approaches;
(iv) topological methods.

We have also included a chapter on nonlinear boundary value problems for functional differential equations and a chapter covering special topics of interest.

The main features of the book are

(i) a coverage of a portion of the material from the contribution of Russian mathematics of which the English speaking world is not well aware;

(ii) the use of several Lyapunov-like functions and differential inequalities in a fruitful way;

(iii) the inclusion of many examples and problems to help the reader develop an expertise in the field.

This book is an outgrowth of a seminar course given by the authors. We have assumed the reader is familiar with the fundamental theory of ordinary differential equations, including the theory of differential inequalities, as well as the basic theory of real and functional analysis. It is designed to serve as a textbook for an advanced course and as a research monograph. It is therefore useful to the specialist and the nonspecialist alike. The reader who is familiar with the contents of the book, it is hoped, is fully equipped to contribute to the area.

Acknowledgments

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Chapter 1
METHODS INVOLVING DIFFERENTIAL INEQUALITIES

1.0 INTRODUCTION

A variety of techniques are employed in the theory of nonlinear boundary value problems. This chapter is primarily concerned with the methods involving differential inequalities. The basic idea is to modify the given boundary value problem suitably, and then to use the theory of differential inequalities and the existence theorems in the small to establish the desired existence results in the large.

After presenting needed existence theorems in the small, we first concentrate on scalar second-order differential equations and associated boundary value problems. We then introduce upper and lower solutions, discuss the modification technique, and utilize Nagumo's condition to obtain a priori bounds on solutions and their derivatives. Once we have these bounds at our disposal, to prove existence theorems on finite or infinite intervals is relatively simple and straightforward. Boundary value problems subjected to nonlinear boundary conditions as well are treated in this framework. We then develop Lyapunov-like theory for boundary value problems employing several Lyapunov-like functions and the theory of differential inequalities in a fruitful way. We also treat in detail Perron's method of proving existence in the large by utilizing the properties of sub- and superfunctions and the existence results in the small. This technique works well for scalar equations.

We next extend the results considered for scalar equations to a finite system of second-order differential equations. Here there are two directions to follow, that is, either try to obtain the required bounds componentwise or in terms of a convenient norm. We offer results from both points of view indicating their relative merits and using Lyapunov-like theory, whenever possible, to derive general results.

1.1 EXISTENCE IN THE SMALL

Let R^n denote the real n-dimensional, Euclidean space and for $x \in R^n$, let $\|x\|$ denote any convenient norm of x. Let J be the interval $[a,b]$. We shall mean by $C^{(n)}[A,B]$ the class of n-times continuously differentiable functions from a set A into a set B.

We will be concerned, in this section, with the existence of solutions of the second-order differential equations of the form

$$x'' = f(t,x,x'), \tag{1.1.1}$$

satisfying the boundary conditions

$$x(t_1) = x_1, \quad x(t_2) = x_2, \quad t_1, t_2 \in J, \tag{1.1.2}$$

where $f \in C[J \times R^n \times R^n, R^n]$. For the purposes of this chapter, we also need an existence result under more general boundary conditions. This we do consider for the scalar case, leaving a thorough discussion of the general theory to a later chapter.

First of all, we observe that the only solution of

$$x'' = 0, \tag{1.1.3}$$

subject to the boundary conditions

$$x(t_1) = 0, \qquad x(t_2) = 0, \tag{1.1.4}$$

is the trivial solution. This implies, from the theory of linear differential equations, that there exists a unique solution of

$$x'' = h(t), \tag{1.1.5}$$

satisfying (1.1.4) for each $h \in C[J,R^n]$. Moreover, since the problem (1.1.3), (1.1.4) possesses the two linearly independent solutions $u(t) = (t - t_1)$, $v(t) = (t_2 - t)$, the method of variation of parameters readily gives the integral equation

$$x(t) = \frac{1}{t_1 - t_2}\left[\int_{t_1}^{t} (t_2 - t)(s - t_1)h(s)\,ds + \int_{t}^{t_2} (t - t_1)(t_2 - s)h(s)\,ds\right] \tag{1.1.6}$$

for the solution $x(t)$ of (1.1.5) subject to (1.1.4). Relation (1.1.6) can be written in the familiar form

$$x(t) = \int_{t_1}^{t_2} G(t,s)h(s)\,ds, \tag{1.1.7}$$

where

$$G(t,s) = \begin{cases} (t_2 - t)(s - t_1)/(t_1 - t_2), & t_1 \le s \le t \le t_2, \\ (t_2 - s)(t - t_1)/(t_1 - t_2), & t_1 \le t \le s \le t_2. \end{cases}$$

This function $G(t,s)$ is usually referred to as the Green's function for the boundary value problem in question. Hence the solution of (1.1.5) verifying conditions (1.1.2) takes the form

$$x(t) = \int_{t_1}^{t_2} G(t,s)h(s)\,ds + w(t), \tag{1.1.8}$$

where $w''(t) = 0$ and $w(t_1) = x_1$, $w(t_2) = x_2$. It therefore follows that if $x(t)$ is a solution of (1.1.1), (1.1.2), then

$$x(t) = \int_{t_1}^{t_2} G(t,s)f\big(s,x(s),\ x'(s)\big)\,ds + w(t). \tag{1.1.9}$$

Conversely, if $x(t)$ is a solution of (1.1.9), we can verify by differentiation of (1.1.6) that $x(t)$ satisfies (1.1.1), (1.1.2).

Let us next recall some properties of the function $G(t,s)$ for later use. For a fixed t, the maximum of $|G(t,s)|$ is attained at $s=t$ and $|G(t,t)|$ has its maximum value at $t = (t_1+t_2)/2$, that is,

$$|G(t,s)| \le (t_2-t_1)/4\,. \tag{1.1.10}$$

Furthermore,

$$\int_{t_1}^{t_2} |G(t,s)|\,ds = (t_2-t)(t-t_1)/2$$

and consequently

$$\int_{t_1}^{t_2} |G(t,s)|\,ds \le (t_2-t_1)^2/8\,. \tag{1.1.11}$$

Moreover,

$$\int_{t_1}^{t_2} |G_t(t,s)|\,ds = \big((t-t_1)^2 + (t_2-t)^2\big)/2(t_2-t_1)\,,$$

the maximum of which is attained at $t=t_1$ and $t=t_2$. Hence, we obtain

$$\int_{t_1}^{t_2} |G_t(t,s)|\, ds \le (t_2 - t_1)/2 . \tag{1.1.12}$$

We are now ready to prove an existence and uniqueness result by using the contraction mapping theorem.

THEOREM 1.1.1. Let $f \in C[J \times R^n \times R^n, R^n]$ and for $(t,x_1,y_1),\ (t,x_2,y_2) \in J \times R^n \times R^n$,

$$\|f(t,x_1,y_1) - f(t,x_2,y_2)\| \le K\|x_1 - x_2\| + L\|y_1 - y_2\|, \tag{1.1.13}$$

where $K,\ L > 0$ are constants such that

$$K\left((t_2 - t_1)^2/8\right) + L\left((t_2 - t_1)/2\right) < 1 . \tag{1.1.14}$$

Then the boundary value problem (1.1.1), (1.1.2) has a unique solution.

Proof: Let B be the Banach space of functions $u \in C^{(1)}\left[[t_1,t_2],\ R^n\right]$ with the norm

$$\|u\|_B = \max_{t_1 \le t \le t_2} \left[K\|u(t)\| + L\|u'(t)\|\right].$$

Define the operator $T\colon B \to B$ by

$$Tu(t) = \int_{t_1}^{t_2} G(t,s) f\big(s,u(s),u'(s)\big)\, ds + w(t).$$

We then have, by (1.1.11) and (1.1.13),

$$\|Tu_2(t) - Tu_1(t)\| \le \frac{(t_2 - t_1)^2}{8}\left[K\|u_2(t) - u_1(t)\| + L\|u_2'(t) - u_1'(t)\|\right]$$

$$\le \frac{(t_2 - t_1)^2}{8}\|u_2 - u_1\|_B .$$

Also, because of (1.1.12),

$$\|Tu_2'(t) - Tu_1'(t)\| \leq \frac{(t_2 - t_1)}{2} \left[K\|u_2(t) - u_1(t)\| + L\|u_2'(t) - u_1'(t)\|\right]$$

$$\leq \frac{(t_2 - t_1)}{2} \|u_2 - u_1\|_B .$$

It then follows that

$$\|Tu_2 - Tu_1\|_B \leq \left[K \frac{(t_2 - t_1)^2}{8} + L \frac{(t_2 - t_1)}{2} \|u_2 - u_1\|_B\right].$$

This, in view of assumption (1.1.14), shows that T is a contraction mapping and thus has a unique fixed point which is the solution of the problem (1.1.1), (1.1.2). The proof is complete.

An interesting problem is to find the largest possible interval in which the preceding theorem is valid. In the case when $f \in C[J \times R \times R, R]$, one can offer such a best possible result. We have intentionally given such a result in the following exercise with generous hints.

EXERCISE 1.1.1. Assume that $f \in C[J \times R \times R, R]$ and satisfies (1.1.13). Let $u(t)$ be any solution of

$$u'' + Lu' + Ku = 0 \tag{1.1.15}$$

which vanishes at $t = t_1$ and let $\alpha(L,K)$ be the first unique number such that $u'(t) = 0$ for $t = t_1 + \alpha(L,K)$. Show that the boundary value problem (1.1.1), (1.1.2) has a unique solution if $t_2 - t_1 < 2\alpha(L,K)$ and that this result is best possible.

Hints: Step 1. First show that there is a unique solution to the boundary value problem $x'' = f(t,x,x')$, $x(t_1) = x_1$, $x'(t_3) = x_3$ if $(t_3 - t_1) < \alpha(L,K)$. This can be shown by applying the contraction mapping theorem relative to the Banach space $E = C^{(1)}\left[[t_1,t_3],R\right]$ with the norm

$$\|v\|_E = \max\left[\max_{t_1 \le t \le t_3} |v(t)|/u_0(t), \max_{t_1 \le t \le t_3} |v'(t)|/u_0'(t) \right],$$

where $u_0(t) > 0$ is a solution of

$$u'' + \frac{Lu' + Ku}{\alpha} = 0$$

for α sufficiently close to 1.

Step 2. Show that the existence of unique solutions of (1.1.1), (1.1.2) and of (1.1.1) with either $x(t_1) = x_1$, $x'(t_3) = x_3$ or $x'(t_1) = x_3$, $x(t_2) = x_2$ on any interval of length less than d implies the existence of a unique solution of (1.1.1), (1.1.2) on any interval of length less than 2d.

Step 3. To prove that the result is the best possible show that $u'' + L|u'| + Ku = 0$ has a nontrivial solution verifying $u(t_1) = u(t_2) = 0$, where $t_2 - t_1 = 2\alpha(L,K)$. Since $u(t) \equiv 0$ also satisfies the problem, argue that the result is best possible. Observe that $\alpha(L,K)$ can be explicitly computed.

EXERCISE 1.1.2. Show that it is sufficient to define f in Theorem 1.1.1 for $t \in [t_1,t_2]$, $\|x\| \le N$, $\|x'\| \le 4N/(t_2 - t_1)$, where N satisfies either

$$m \frac{(t_2 - t_1)^2}{8} \le N \left[1 - \left(K \frac{(t_2 - t_1)^2}{8} + L \frac{(t_2 - t_1)}{2} \right) \right],$$

if $m = \max_{t_1 \le t \le t_2} \|f(t,0,0)\|$, or $M\,(t_2 - t_1)^2/8 \le N$, if $M = \max \|f(t,x,x')\|$ for $t \in [t_1,t_2]$, $\|x\| \le N$, $\|x'\| \le 4N/(t_2 - t_1)$.

<u>Hint</u>: Apply the contraction mapping theorem on the ball $\|u\|_0 \le N$ where

$$\|u\|_0 = \max\left[\max_{t_1 \le t \le t_2} \|u(t)\|, \frac{(t_2 - t_1)}{4} \max_{t_1 \le t \le t_2} \|u'(t)\|\right].$$

To obtain merely an existence result we employ Schauder's fixed point theorem as is usual.

THEOREM 1.1.2. Let $M > 0$, $N > 0$ be given numbers and let, for $t \in J$,

$$\|x\| \le 2M, \qquad \|y\| \le 2N, \qquad \|f(t,x,y)\| \le q,$$

where $f \in C[J \times R^n \times R^n, R^n]$. Suppose that $\delta = \min[(8M/q)^{\frac{1}{2}}, 2N/q]$. Then any problem (1.1.1), (1.1.2) such that $[t_1,t_2] \subset J$, $t_2 - t_1 \le \delta$, $\|x_1\| \le M$, $\|x_2\| \le M$, and $\|x_1 - x_2\|/(t_2 - t_1) \le N$, has a solution. Furthermore, given any $\varepsilon > 0$, there is a solution $x(t)$ such that $\|x(t) - w(t)\| < \varepsilon$, $\|x'(t) - w'(t)\| < \varepsilon$ on $[t_1,t_2]$, provided $t_2 - t_1$ is sufficiently small.

Proof: Consider the Banach space $B = C^{(1)}\left[[t_1,t_2],R^n\right]$ with the norm $\|x\|_B = \max_{t_1 \le t \le t_2} \|x(t)\| + \max_{t_1 \le t \le t_2} \|x'(t)\|$. Notice that the set

$$B_0 = [x \in B: \|x\| \le 2M, \|x'\| \le 2N]$$

is a closed, convex subset of B. Define the mapping $T: B \to B$ by

$$Tx(t) = \int_{t_1}^{t_2} G(t,s)\, f\big(s,x(s),x'(s)\big)\, ds + w(t).$$

Using now estimates (1.1.11), (1.1.12), we obtain

$$\|Tx(t)\| \le \big((t_2 - t_1)^2/8\big)\, q + M$$