THE THEORY OF GAUGE FIELDS IN FOUR DIMENSIONS

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by

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Preface

These notes result from a meeting held in Santa Barbara in August 1983. The purpose of the meeting was to bring together geometric topologists and differential geometers to study in depth the recent work of Simon Donaldson. Of course, due to the beautiful and profound results of Mike Freedman, the subject of 4-manifolds had already become the focus of lively interest. Moreover, in light of Donaldson's result, the Freedman–Casson machinery was able to produce the startling fact that there exist exotic differentiable structures on $\mathbb{R}^4$. For these reasons topologists have wanted to understand in depth the arguments of Donaldson, which are based on the theory of Yang–Mills fields.

Consequently, the principal purpose of these lectures (and these notes) is to present these arguments together with all the background material required by someone who is not an expert in the field. The lectures are aimed, however, at mature mathematicians with some training in geometry and topology. The task set out here was already sufficiently exacting that no time was available for wide-ranging discussion or excursions into physics. On the other hand, this presentation attempts to be nearly complete.

It should be mentioned that a seminar on this subject was run last spring at M.S.R.I. by M. Freedman and K. Uhlenbeck. The notes of this seminar have been prepared for publication with the assistance of D. Freed. They also provide a detailed reference for this material.

The success of the conference was due to the enormous efforts of the organizing committee: Ken Millett, Doug Moore, and Marty Scharlemann, to whom all of us who participated have expressed our gratitude. I would also like to thank Susan Crofoot for her beautiful job of preparing the manuscript and the organizing committee for all the help they gave me with proofreading.

It is a pleasure to report that two of the conference participants, Ron Fintushel and Ron Stern, have subsequently succeeded in greatly generalizing the results of Donaldson while, at the same time, simplifying the arguments. They have also applied their methods to prove a spectacular result concerning homology cobordisms of homology 3-spheres. In particular, they show that the Poincaré 3-sphere has infinite order in this group. Interestingly, the key to their arguments is to consider gauge fields with SO_3 in place of SU_2. For low instanton numbers, the moduli space of self-dual connections in this case is actually *compact*. We shall say a bit more about this at the end of Chapter I.

Contents

I. Introduction

It is always a wonderful event in mathematics when the results of one discipline of thought have startling and unexpected consequences in another. This was recently the case when Simon Donaldson, arguing from deep results in gauge field theory, proved the nonexistence of differentiable structures on certain compact 4-manifolds. The result was timely, for, in what must be considered one of the ultimate achievements of topology, Mike Freedman had recently given a complete classification of compact topological 4-manifolds (in the simply-connected case.) In fact, as Freedman and Kirby first observed, this theory, together with Donaldson's result, implies the existence of exotic differentiable structures on $\mathbf{R}^4$.

The purpose of these lectures is to present Donaldson's theorem together with the foundational work in gauge field theory, due to Uhlenbeck, Taubes, Atiyah, Hitchin, Singer, and others, on which the arguments are based. This first chapter is an introduction. We begin by summarizing the current state of affairs in the theory of 4-manifolds. We then state Donaldson's theorem and give a brief outline of its proof.

1. Connected surfaces. One of the classical results of topology is the classification of compact connected surfaces (without boundary) up to diffeomorphism. The result can be presented in the following way. Let Σ be such a surface and consider closed curves γ_1 and γ_2 on Σ. By a small deformation we can make γ_1 transversal to γ_2. The curves then intersect in a finite number of points. This number, modulo 2, turns out to depend only on the homology class of γ_1 and γ_2 in $H_1(\Sigma; \mathbf{Z}_2)$. We thereby get a symmetric bilinear form μ on $H_1(\Sigma; \mathbf{Z}_2)$ called the *intersection form* of Σ. Poincaré duality says that this form is nondegenerate. We define the form to be of *type* II if $\mu(x, x) = 0$ for all x; otherwise, we say it is of *type* I.

Note that if $\gamma \subset \Sigma$ is an embedded curve along which orientation is reversed, then a tubular neighborhood of γ is a Möbius band and $\mu([\gamma], [\gamma]) = 1$ (see Figure 1). Hence, μ is of type I.

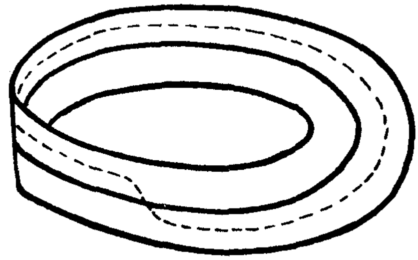

FIGURE 1

THEOREM 1.1. *Two compact connected surfaces are diffeomorphic if and only if their intersection forms are abstractly equivalent. Surfaces can be separated into two classes: type* I *and type* II. *Those of type* I *are nonorientable and can be decomposed into a connected sum of real projective planes. Those of type* II *are orientable and can be decomposed into a connected sum of tori.*

The relevant data can be organized as shown in Table 1.

Type I	nonorientable ($w_1 \neq 0$)	$\mu \cong \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$	$\Sigma \cong \mathbf{P}^2(\mathbf{R})\# \cdots \#\mathbf{P}^2(\mathbf{R})$
Type II	orientable ($w_1 = 0$)	$\mu \cong \begin{pmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & & \\ & \ddots & \\ & & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{pmatrix}$	$\Sigma \cong (S^1 \times S^1)\# \cdots \#(S^1 \times S^1)$

TABLE 1

2. Simply-connected 4-manifolds. One of the glorious achievements of modern topology is the recent classification of compact simply-connected topological 4-manifolds. As with surfaces, the classification is stated in terms of an intersection form on the middle-dimensional homology group. Now a simply-connected manifold M^4 can be oriented. Therefore, using the orientations, the intersection of two transversal, oriented surfaces can be counted as an integer. This gives a symmetric bilinear form μ on $H_2(M; \mathbf{Z})$. Poincaré duality states that this form is *unimodular*. That is, if μ is expressed as an $(r \times r)$-matrix $((m_{ij}))$ with integer entries (with respect to some basis of the free abelian group $H_2(M; \mathbf{Z})$), then $\det((m_{ij})) = \pm 1$. It is a classical result that the form μ determines M up to homotopy type.

THEOREM 2.1 (J. H. C. WHITEHEAD (1949) [Wh]). *Two compact simply-connected 4-manifolds are homotopy equivalent if and only if their intersection forms are equivalent.*

(For a nice proof see [MH, p. 103].)

Recall that two symmetric bilinear forms μ_1 and μ_2 on lattices (i.e., finitely generated free abelian groups) Λ_1 and Λ_2, respectively, are ***equivalent*** if there is an isomorphism $\phi: \Lambda_1 \to \Lambda_2$ such that $\phi^*\mu_2 = \mu_1$. Such a form on a lattice Λ is of *type* II if $\mu(x, x) \equiv 0 \pmod 2$ for all $x \in \Lambda$. Otherwise, μ is of *type* I.

There are two fundamental invariants of a symmetric bilinear form μ on a lattice Λ: its *rank* (the dimension of $\Lambda \otimes \mathbf{R}$) and its *signature* ($=$ rank $- 2q$, where q is the maximal dimension of a subspace of $\Lambda \otimes \mathbf{R}$ on which μ is negative definite). It is an elementary result that the signature of a form of type II must be a multiple of 8.

Indefinite unimodular symmetric bilinear forms are completely determined up to equivalence by their rank and signature. The classification is as follows.

(2.2) Type I: $\mu \cong \langle 1 \rangle \oplus \cdots \oplus \langle 1 \rangle \oplus \langle -1 \rangle \oplus \cdots \oplus \langle -1 \rangle,$

(2.3) Type II: $\mu \cong H \oplus \cdots \oplus H \oplus E_8 \oplus \cdots \oplus E_8,$

where $\langle 1 \rangle$ and $\langle -1 \rangle$ denote the two possible rank-1 forms,

(2.4)
$$H \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad E_8 \equiv \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

Note that the nonzero off-diagonal elements of E_8 can be associated to the Dynkin diagram

for the exceptional Lie group E_8. Note that indefiniteness forces both $\langle 1 \rangle$ and $\langle -1 \rangle$ to appear in (2.2) and at least one H to appear in (2.3).

Definite unimodular symmetric bilinear forms are a different matter altogether. Their study forms one of the difficult, classical fields of mathematics. To illustrate this, we present the following astonishing table. Let $N(r)$ denote the number of inequivalent unimodular type II forms which are positive definite and of rank r.

(2.5)

r	8	16	24	32	40
$N(r)$	1	2	24	$\geq 10^7$	$\geq 10^{51}$

TABLE 2

(As a basic reference for the above facts, see Milnor–Husemoller [MH].)

The Whitehead Theorem 2.1 naturally suggests the question:

(2.6) Which forms can appear as intersection forms on compact simply-connected 4-manifolds?

This is an existence question. There is also a uniqueness question.

(2.7) How many inequivalent manifolds can carry the same form?

These questions can be asked for topological manifolds, where "equivalence" means homeomorphism, or for differentiable manifolds, where "equivalence" means diffeomorphism. For the trivial form of rank zero, Question (2.7) is the 4-dimensional Poincaré Conjecture.

It is one of the profound results of modern topology that gives a complete answer to this question in the topological case. As with surfaces, we divide the simply-connected 4-manifolds into two classes: those of *type* I which are nonspin and have intersection forms of type I, and those of *type* II which are spin and have intersection forms of type II. Let $\mathscr{M}_R^{\mathrm{TOP}}$ denote the homeomorphism classes of compact oriented simply-connected topological 4-manifolds of type R. Let $\mathscr{I}_R$ denote the equivalence classes of unimodular symmetric bilinear forms of type R. Taking the intersection form of a manifold gives a map $i_R\colon \mathscr{M}_R^{\mathrm{TOP}} \to \mathscr{I}_R$.

THEOREM 2.8 (M. FREEDMAN (1982) [F]). *The map* $i_{\mathrm{II}}\colon \mathscr{M}_{\mathrm{II}}^{\mathrm{TOP}} \to \mathscr{I}_{\mathrm{II}}$ *is a bijection. The map* $i_{\mathrm{I}}\colon \mathscr{M}_{\mathrm{I}}^{\mathrm{TOP}} \to \mathscr{I}_{\mathrm{I}}$ *is exactly two-to-one and onto.*

Thus, every unimodular form is the intersection form of a simply-connected topological 4-manifold. This manifold is unique in the type II case, and there are exactly two distinct manifolds in the type I case. The two possibilities differ in that one of them has a nonzero Kirby–Siebenmann obstruction to triangulability (see [KS]).

3. Differentiable 4-manifolds. We now examine the situation for the differentiable case. It has been known for some time that not every form can appear on a differentiable 4-manifold.

THEOREM 3.1 (ROCHLIN [Ro]). *Let M be a compact simply-connected differentiable 4-manifold of type* II. *Then* signature$(M) \equiv 0 \pmod{16}$.

Recall that the signature of a type II form is always a multiple of 8. However, forms of type II with signature 8 do exist—for example, the form E_8 above. Thus, for a unimodular form μ of type II, we are led to consider the following *Rochlin invariant* $\rho(\mu) \equiv \frac{1}{8}$ signature$(\mu) \pmod 2$. Forms with nonzero Rochlin invariant do not occur as intersection forms on compact oriented smooth 4-manifolds. Until recently, little else was known about this question.

We now come to the theorem whose proof is the main focus of these lectures.

THEOREM 3.2 (S. DONALDSON (1982) [$\mathrm{D}_{1,2}$]). *Let M be a compact simply-connected smooth 4-manifold whose intersection form μ is positive definite. Then μ is equivalent to the diagonal form; i.e.,* $\mu \cong \langle 1\rangle \oplus \cdots \oplus \langle 1\rangle$.

Thus, the inpenetrable jungle of positive definite symmetric bilinear forms is removed from the question of understanding smooth 4-manifolds. Using Freedman's theorem we obtain a simple list (see Table 3) of the ***homeomorphism*** classes of simply-connected smooth 4-manifolds which looks remarkably similar to the one given for connected smooth 2-manifolds.

Type I	not spin ($w_2 \neq 0$)	$\mu = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$	$M = \mathbf{P}^2(\mathbf{C}) \# \cdots \# \mathbf{P}^2(\mathbf{C}) \# \overline{\mathbf{P}^2}(\mathbf{C}) \# \cdots \# \overline{\mathbf{P}^2}(\mathbf{C})$
Type II	spin ($w_2 = 0$)	$\mu = \begin{pmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & & & \\ & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & & \\ & & E_8 & \\ & & & E_8 \end{pmatrix}$	$M = (S^2 \times S^2) \# \cdots \# (S^2 \times S^2) \# E_8 \# \cdots \# E_8$

TABLE 3

In the second row there must be at least one $S^2 \times S^2$ and an even number of E_8 factors (by 3.1). With these restrictions it is still unknown which connected sums of $(S^2 \times S^2)$'s and E_8's can be smoothed. (The classical $K3$ surface has three $(S^2 \times S^2)$'s and two E_8's.)

4. Exotica. At this moment the uniqueness of differentiable structures on compact 4-manifolds remains an open question. However, the results discussed above combine to give the following rather startling fact, first observed by M. Freedman and R. Kirby.

THEOREM 4.1. *There exists an exotic* $\mathbf{R}^4$*—that is, a manifold homeomorphic to, but not diffeomorphic to,* $\mathbf{R}^4$.

The original proof of this fact used the basics of Freedman's theorem. However, a slightly different and very pretty approach to the question is given by using a result of F. Quinn. Recall that the complex projective plane $\mathbf{P}^2(\mathbf{C})$ is obtained by adding a "line at infinity" to $\mathbf{C}^2$. In particular, if $S^2 = \mathbf{P}^1(\mathbf{C}) \subset \mathbf{P}^2(\mathbf{C})$ is any projective line, then

$$\mathbf{P}^2(\mathbf{C}) - S^2 = \mathbf{R}^4.$$

Let h: $\mathbf{P}^2(\mathbf{C}) \to \mathbf{P}^2(\mathbf{C})$ be a homeomorphism. Then the set $\mathbf{P}^2(\mathbf{C}) - h(S^2) = h(\mathbf{R}^4)$ is clearly homeomorphic to $\mathbf{R}^4$ and inherits a differentiable structure as an open subset of $\mathbf{P}^2(\mathbf{C})$.

Suppose now that M is a 4-manifold with a differentiable structure defined outside of some point p.

DEFINITION. The singular point p is *resolvable* if there is a homeomorphism h: $\mathbf{P}^2(\mathbf{C}) \to \mathbf{P}^2(\mathbf{C})$ and a neighborhood U of p so that $U - \{p\}$ is diffeomorphic to $\tilde{U} - h(S^2)$ for some neighborhood $\tilde{U}$ of $h(S^2)$ in $\mathbf{P}^2(\mathbf{C})$.

Roughly speaking, the point p is resolvable if the smooth structure can be extended across when p is replaced by a 2-sphere of self-intersection ± 1. This is very much like the "blowing-up" process in algebraic geometry.

THEOREM 4.2 (F. QUINN [Q]). *Let M be a compact topological 4-manifold whose Kirby–Siebenmann invariant is zero. Then M has a smooth structure defined outside a finite set of points with the property that each singular point is resolvable.*

Consider now a 4-manifold with $KS = 0$, but which, by Donaldson's theorem, is still not smoothable. The manifold $M = E_8 \# \mathbf{P}^2(\mathbf{C})$ will do. By Quinn M carries an almost smooth structure with a finite number of resolvable singular points $p_1, \ldots, p_n$. That is, each p_j has a neighborhood U_j so that $U_j' \equiv U_j - \{p_j\}$ is diffeomorphic to $\tilde{U}_j' \equiv \tilde{U}_j - h_j(S^2)$ for some $\tilde{U}_j$, h_j as above. Note that $\tilde{U}_j'$ is a neighborhood of infinity for some differentiable structure on $\mathbf{R}^4 \approx \tilde{\mathbf{R}}_j^4 = \mathbf{P}^2(\mathbf{C}) - h_j(S^2)$. Therefore we say that the smooth manifold $M - \{p_1, \ldots, p_n\}$ has *euclidean ends*. See Figure 2.

Claim. At least one of the open differentiable manifolds $\tilde{\mathbf{R}}_j^4 \equiv \mathbf{P}^2(\mathbf{C}) - h_j(S^2)$ is not diffeomorphic to $\mathbf{R}^4$.

This follows immediately from any of the following assertions.

ASSERTION A. At least one of the manifolds $\tilde{\mathbf{R}}_j^4$ contains a compact set which cannot be surrounded by a smoothly embedded S^3 (i.e., so the set is contained in the bounded component of $\tilde{\mathbf{R}}_j^4 - S^3$).

ASSERTION B. At least one of the $\tilde{\mathbf{R}}_j^4$'s does not admit an orientation-reversing diffeomorphism.

ASSERTION C. At least one of the $\tilde{\mathbf{R}}_j^4$'s has the property that it admits no orientation-preserving embedding into $\mathbf{P}^2(\mathbf{C})$. In fact, it admits no such embedding into any smooth oriented simply-connected 4-manifold with a positive definite intersection form.

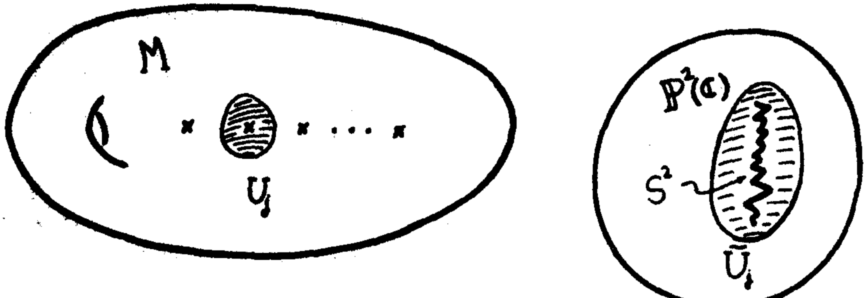

FIGURE 2

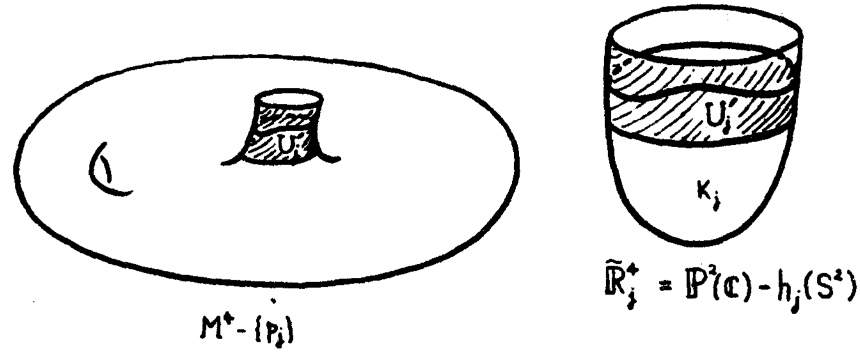

FIGURE 3

PROOF OF A. Let $K_j = \tilde{\mathbf{R}}_j^4 - \tilde{U}_j'$. If each K_j can be surrounded by a smoothly embedded $S^3 \subset \tilde{U}_j'$, then, via the diffeomorphism, we get a smoothly embedded $S^3 \subset U_j'$. Cutting along S^3 and attaching a D^4 gives a smoothing of M, which, by Donaldson, is impossible, see Figure 3.

PROOF OF B. Replacing each U_j by $\tilde{U}_j$ gives a smooth structure on $M' \cong M\#\mathbf{P}^2(\mathbf{C})\# \cdots \#\mathbf{P}^2(\mathbf{C})\#\overline{\mathbf{P}^2(\mathbf{C})}\# \cdots \#\overline{\mathbf{P}^2(\mathbf{C})}$, where $\overline{\mathbf{P}^2(\mathbf{C})}$ denotes $\mathbf{P}^2(\mathbf{C})$ with the opposite orientation. (Note that the orientation of $\mathbf{P}^2(\mathbf{C})$ is detected by the self-intersection number of the S^2.) By Donaldson there must appear at least one $\overline{\mathbf{P}^2(\mathbf{C})}$, for otherwise M' would be smoothable with self-intersection form $E_8 \oplus \langle 1 \rangle \oplus \cdots \oplus \langle 1 \rangle$, which is positive definite and nontrivial. However, if each $\tilde{\mathbf{R}}_j^4$ carried an orientation-reversing diffeomorphism, we could reverse the gluing of the $\overline{\mathbf{P}^2(\mathbf{C})}$-factors to make them positive, i.e., $\mathbf{P}^2(\mathbf{C})$-factors. As noted, this is not possible. □

PROOF OF C. Suppose there exists an *orientation-preserving* smooth embedding $i_j\colon \tilde{\mathbf{R}}_j^4 \hookrightarrow \mathbf{P}^2(\mathbf{C})$. If we remove $i_j(K_j) = i_j(\tilde{\mathbf{R}}_j^4 - \tilde{U}_j')$ from $\mathbf{P}^2(\mathbf{C})$, we obtain a smooth manifold, homeomorphic to $\mathbf{P}^2(\mathbf{C}) - \{\mathrm{pt}\}$, whose end is oriented diffeomorphic to $\tilde{U}_j' \cong U_j'$. We can then attach $\mathbf{P}^2(\mathbf{C}) - i_j(K_j)$ smoothly to $M - \{p_j\}$. If this works for all j, we get a smooth version of $E_8\#\mathbf{P}^2(\mathbf{C})\# \cdots \#\mathbf{P}^2(\mathbf{C})$ as before. See Figure 4. □

R. Gompf has shown that there exists a smooth structure on $M - \{p\}$ (where $M = E_8\#\mathbf{P}^2(\mathbf{C})$) so that the singular point p is resolvable. By the above remarks we thereby have a euclidean end with all of the above character traits.

THEOREM 4.3 (R. GOMPF [G]). *There exists an exotic $\mathbf{R}^4$, denoted $\mathbf{R}_\Gamma^4$, with the properties*:

(i) *$\mathbf{R}_\Gamma^4$ contains a compact set which cannot be surrounded by a smoothly embedded 3-sphere.*

(ii) $\mathbf{R}^4_\Gamma$ *admits no orientation-reversing diffeomorphisms.*

(iii) $\mathbf{R}^4_\Gamma$ *has no oriented smooth embedding into* $\mathbf{P}^2(\mathbf{C})$ *or* $\mathbf{P}^2(\mathbf{C})\# \cdots \#\mathbf{P}^2(\mathbf{C})$ *(for any number of factors).*

(iv) $\mathbf{R}^4_\Gamma$ *has an end which appears on a smoothing of* $E_8\#\mathbf{P}^2(\mathbf{C})$.

This creature $\mathbf{R}^4_\Gamma$ can be used to generate further examples. There is a simple connected sum operation for open manifolds given by building an open bridge between them. More rigorously, one takes properly embedded arcs $\lambda_j: [0, \infty) \to M_j$ for $j = 1, 2$. Each arc has a tubular neighborhood U_j which is diffeomorphic to $(0,1) \times \mathbf{R}^3$. Attach $(0,3) \times \mathbf{R}^3$ to $M_1 \amalg M_2$ by gluing $(0,1) \times \mathbf{R}^3$ to U_1 and $(2,3) \times \mathbf{R}^3$ to U_2. We shall denote the resulting connected sum by $M_1 \natural M_2$. See Figure 5.

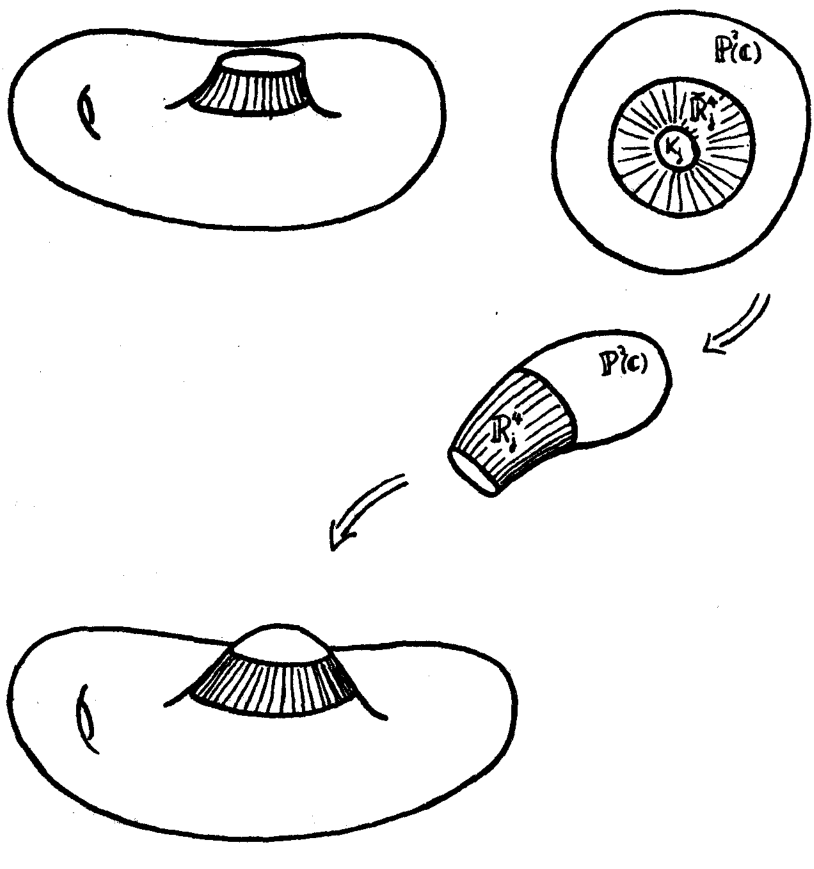

FIGURE 4

FIGURE 5

Caution. It is not clear that this construction is independent of the choice of arcs and attaching maps. Indeed, the attaching maps should be made "canonically" in thin regular neighborhoods of the curves (using the linearization of the manifold along the curve in the spirit of the classical case.) It is left as an exercise to the reader to verify, via isotopy of curves and uniqueness of regular neighborhoods, that the operation $\natural$ is well defined.

By building the above bridge in a symmetric fashion (like "doubling"), we can take a connected sum

$$\mathbf{R}^4_S = \mathbf{R}^4_\Gamma \natural (-\mathbf{R}^4_\Gamma)$$

which does have an orientation-reversing diffeomorphism. Since $\mathbf{R}^4_S$ contains $\pm\mathbf{R}^4_\Gamma$, it cannot be embedded into any smooth simply-connected 4-manifolds with either a positive or negative definite intersection form (by Theorem 4.3). In particular, $\mathbf{R}^4_S$ is exotic and not diffeomorphic to $\mathbf{R}^4_\Gamma$.

These two spaces can be used for many interesting constructions. Let $\mathbf{R}^4_E$ be any exotic $\mathbf{R}^4$. By embedding two arcs in $\mathbf{R}^4_E$ as above and pasting on copies of $(0,3) \times \mathbf{R}^3$ along $(0,1) \times \mathbf{R}^3$, we can create an $\hat{\mathbf{R}}^4_E$ with two "trivialized projecting ends." (See Figure 6.) We can now concatenate these fellows in infinite arrays. For example, let $w = \cdots gsggs \cdots$ be any doubly infinite word in two letters g and s. Then we can join together the blocks $\hat{\mathbf{R}}^4_\Gamma$ and $\hat{\mathbf{R}}^4_S$ to give a manifold

$$\mathbf{R}^4_w = \cdots \hat{\mathbf{R}}^4_\Gamma \natural \hat{\mathbf{R}}^4_S \natural \hat{\mathbf{R}}^4_\Gamma \natural \hat{\mathbf{R}}^4_\Gamma \natural \hat{\mathbf{R}}^4_S \cdots.$$

Here, $\natural$ is the canonical operation on the trivialized projecting ends. These creatures are all exotic since they contain $\mathbf{R}^4_\Gamma$ as an open subset. It is interesting to speculate how many distinct $\mathbf{R}^4$'s can be constructed this way.

Observe now that these basic building blocks $\hat{\mathbf{R}}^4_E$ can be arranged into the pattern of any (countable) tree. A good one to use here is $\hat{\mathbf{R}}_S$, which can be

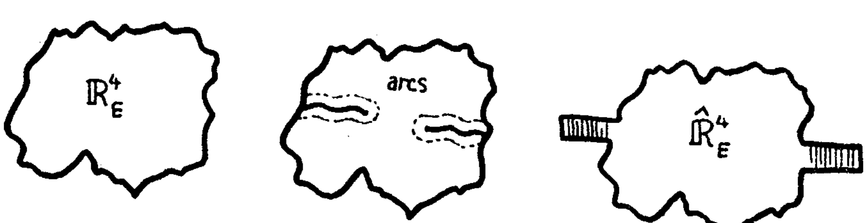

FIGURE 6

assumed to have a reflection symmetry that interchanges the ends. Let T be a countable tree and build an exotic $\mathbf{R}^4$ by joining together copies of $\tilde{\mathbf{R}}^4_S$ as though they were the edges of the tree (see Figure 7). The resulting $\mathbf{R}^4_T$ will contain a group of diffeomorphisms $\mathscr{A}(T) \subset \mathrm{Diff}(\mathbf{R}^4_T)$ isomorphic to the group of automorphisms of T such that for any compact set $K \subset \mathbf{R}^4_T$, the set $\{g \in \mathscr{A}(T): g(K) \cap K\}$ is finite. It seems possible that no two elements in $\mathscr{A}(T)$ are isotopic, so that $\mathscr{A}(T)$ is a subgroup of the component group $\mathrm{Diff}(\mathbf{R}^4_T)/\mathrm{Diff}^0(\mathbf{R}^4_T)$. (Here $\mathrm{Diff}^0(\mathbf{R}^4_T)$ denotes the connected component of the identity.)

Of course, highly symmetric trees do exist. A good example is the universal cover of $S^1 \vee \cdots \vee S^1$ (k times), whose automorphism group contains the free group on k generators and the symmetric group on $2k$ elements.

It should be noted that 4 is the only dimension in which euclidean space carries exotic differentiable structures (see [KS]). Thus, for any exotic $\mathbf{R}^4_E$ we have a diffeomorphism $\mathbf{R}^4_E \times \mathbf{R} \cong \mathbf{R}^5$.

5. An introduction to Donaldson's proof. Many nontrivial results in the topology of 2-manifolds are proved by using complex analysis. A riemannian metric is introduced on the manifold, which, in turn, determines a conformal structure. Using this conformal structure, one then does analysis. A good example is the following. Suppose Σ is a compact simply-connected surface. Then appropriate use of the Riemann–Roch theorem asserts that there is a meromorphic function on Σ of degree 1. That is, there exists a holomorphic map $f: \Sigma \to S^2$ of degree 1. From our knowledge of the singularities of holomorphic maps (they are branched coverings), we conclude that f is a diffeomorphism.

Donaldson's argument is very much in this spirit. Given a compact simply-connected 4-manifold M, we introduce on M a riemannian metric. We then proceed to study a global system of first-order differential equations, much like the Cauchy–Riemann equations. This system depends only on the conformal structure determined by the metric. A detailed understanding of the solutions will have tremendous implications for the global topology of M.

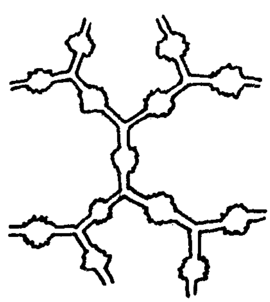

FIGURE 7

In doing complex analysis on compact (Riemann) surface Σ, one is led inevitably to study the complex line bundles on Σ. Let $\mathbb{L}_{\mathbf{C}}(\Sigma)$ denote the topological equivalence classes of such bundles. Then there are bijections

$$(5.1) \qquad \mathbb{L}_{\mathbf{C}}(\Sigma) \overset{1\text{-}1}{\longleftrightarrow} [\Sigma, S^2] \overset{1\text{-}1}{\longleftrightarrow} H^2(\Sigma; \mathbf{Z}) \cong \mathbf{Z},$$

where $[\Sigma, S^2]$ denotes homotopy classes of maps from Σ to S^2. The isomorphism $\mathbb{L}_{\mathbf{C}}(\Sigma) \to H^2(\Sigma, \mathbf{Z})$ is given by taking the Euler class (or, equivalently, the first Chern class). The fundamental line bundle over S^2, from which all others are induced, is the spinor bundle, whose principal S^1-bundle is just the Hopf fibration $S^3 \to S^2$. It can also be viewed as the tautological line bundle over $\mathbf{P}^1(\mathbf{C}) = S^2$.

To study 4-manifolds we are led, in strict analogy, to study quaternion line bundles. Let $\mathbb{L}_{\mathbf{H}}(M)$ denote the topological equivalence classes of quaternion line bundles over a compact 4-manifold M. Then there are analogous bijections

$$(5.2) \qquad \mathbb{L}_{\mathbf{H}}(M) \overset{1\text{-}1}{\longleftrightarrow} [M, S^4] \overset{1\text{-}1}{\longleftrightarrow} H^4(M; \mathbf{Z}) \cong \mathbf{Z}.$$

The map $\mathbb{L}_{\mathbf{H}}(M) \to H^4(M; \mathbf{Z})$ is again given by the Euler class (or, equivalently, by c_2 or $-p_1/2$). The fundamental line bundle over S^4 is the spinor bundle, whose associated principal S^3-bundle is the Hopf fibration $S^7 \to S^4$. It can also be viewed as the tautological line bundle over $\mathbf{P}^1(\mathbf{H}) = S^4$.

Let M be a compact simply-connected oriented smooth 4-manifold and consider the fundamental quaternion line bundle $E \to M$ with Euler class -1. Introduce a riemannian metric on M and a bundle metric in E which is $\mathbf{H}$-compatible, i.e., which is preserved under scalar multiplication by unit quaternions. We now consider the space $\mathscr{C}$ of $\mathbf{H}$-connections on E which preserve the metric. Each connection $\nabla \in \mathscr{C}$ has an associated *curvature tensor* R^∇ which is an exterior 2-form with values in the bundle $\mathrm{Hom}_{\mathbf{H}}(E, E)$. The riemannian metric on M gives a linear involution on 2-forms called the Hodge $*$-operator, $*: \Lambda^2 \to \Lambda^2$. It depends only on the conformal class of the metric. The 2-forms decompose into $+1$ and -1 eigenspaces under $*$. We shall look for connections ∇ which satisfy the equations

$$(5.3) \qquad *R^\nabla = R^\nabla.$$

Such connections are called *self-dual*. Equations (5.3) are like the Cauchy–Riemann equations. They imply that R^∇ is "harmonic." They also imply that ∇ absolutely minimizes the Yang–Mills action

$$(5.4) \qquad \mathscr{YM}(\nabla) = \tfrac{1}{2}\int_M \|R^\nabla\|^2$$

(which is also conformally invariant).

Let $\mathscr{M}$ denote the set of equivalence classes of self-dual connections on E. The following two results are proved by Donaldson, but rely heavily, as we shall see, on work of Taubes, Uhlenbeck, and Atiyah–Hitchin–Singer. In both theorems we assume that *the intersection form of M is (positive) definite*.