SET THEORY AND ITS LOGIC

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THE BELKNAP PRESS
OF HARVARD UNIVERSITY PRESS
CAMBRIDGE, MASSACHUSETTS · 1963

Distributed in Great Britain by Oxford University Press, London
Library of Congress Catalog Card Number 63-17209
Printed in the United States of America

PREFACE

A preface is not, in my book, an introduction. Somewhere, readers already acquainted with the subject need to be told succinctly what a book covers and how. The book itself gives this information the long way, explaining concepts and justifying assertions. The preface gives it the short way, presupposing some command of the technical terms and concepts.

As explained in the Introduction, the book assumes no prior knowledge of set theory but some of logic. Chapter I, entitled "Logic," builds upon this prerequisite. Mainly what it builds is what I have elsewhere called the virtual theory of classes and relations: a partial counterfeit of set theory fashioned purely of logic. This serves later chapters in two ways. For one thing, the virtual theory affords a useful contrast with the later, real theory. The contrast helps to bring out what the genuine assuming of classes amounts to; what power real classes confer that the counterfeits do not. For another thing, the virtual theory eventually gets merged with the real theory in such a way as to produce a combination which, though not strictly more powerful than the real theory alone would be, is smoother in its running.

This departure last mentioned is one of several that are meant for the theoretician's eye. But at the same time the book is meant to provide a general introduction to staple topics of abstract set theory, and, in the end, a somewhat organized view of the best-known axiomatizations of the subject. Paradoxically, the very novelties of the approach are in part devices for neutralizing idiosyncrasy.

Because the axiomatic systems of set theory in the literature are largely incompatible with one another and no one of them clearly deserves to be singled out as standard, it seems prudent to teach a panorama of alternatives. This can encourage research that may some day issue in a set theory that is clearly best. But the writer who would pursue this liberal policy has his problems. He cannot very well begin by offering the panoramic view, for the beginning reader will appreciate neither the material that the various systems are meant to organize nor the considerations that could favor one system in any respect over another. Better to begin by orienting the reader with a preliminary informal survey of the subject matter. But here again there is trouble. If such a survey is to get beyond trivialities, it must resort to serious and sophisticated reasoning such as could quickly veer into the antinomies and so discredit itself if not shunted off them in one of two ways: by abandoning the informal approach in favor of the axiomatic after all, or just by slyly diverting the reader's attention from dangerous questions until the informal orientation is accomplished. The latter course calls for artistry of a kind that is distasteful to a science teacher, and anyway it is powerless with readers who hear about the antinomies from someone else. Once they have heard about them, they can no longer submit to the discipline of complex informal arguments in abstract set theory; for they can no longer tell which intuitive arguments count. It is not for nothing, after all, that set theorists resort to the axiomatic method. Intuition here *is* bankrupt, and to keep the reader innocent of this fact through half a book is a sorry business even when it can be done.

In this book I handle the problem by hewing a formal line from the outset, but keeping my axioms weak and reasonable and thus nearly neutral. I postpone as best I can the topics that depend on stronger axioms; and, when these topics have to be faced, I still postpone the stronger axioms by incorporating necessary assumptions rather as explicit hypotheses into the theorems that require them. In this way I manage to introduce the reader at some length to the substance of set theory without any grave breach of neutrality, and yet also without resorting to studied informality or artificial protraction of innocence. After ten such chapters I find myself in a position to present and compare, in the four concluding chapters, a multiplicity of mutually incompatible axiomatizations of the material with which the reader has been familiarized.

More specifically, the weak axioms that thus govern the main body of the work are such as to imply the existence of none but finite classes. Moreover, they do not even postulate any infinite classes hypothetically. To see what I mean by this, consider, in contrast to my axioms, a pair of axioms providing for the existence of the null class Λ and of $x \cup \{y\}$ for all x and y. These axioms, like mine, imply the existence only of finite classes. But, unlike mine, these provide also that *if* an infinite class x exists then so does the further infinite class $x \cup \{y\}$ for any y.

My axioms do provide for the existence of all finite classes of whatever things there are. They are consequently not altogether neutral toward the systems in the literature. They conflict with those systems, such as von Neumann's, in which some classes—"ultimate" classes, I call them—are counted incapable of being members of classes at all. Though one such system was that of my own *Mathematical Logic*, in the present book I defend the finite classes against the ultimate classes.

My axioms of finite classes are enough, it turns out, for the arithmetic of the natural numbers. Usually the definition of natural number involves infinite classes. Natural numbers are the common members of all classes that contain 0 (somehow defined) and are closed with respect to the successor operation (somehow defined); and any such class is infinite. The law of mathematical induction, based on this definition, can be proved only by assuming infinite classes. But I get by with finite classes by inverting the definition of natural number, thus: x is a natural number if 0 is a common member of all classes that contain x and are closed with respect to predecessor.

Classically the definition of natural number is a special case of that of Frege's ancestral (Dedekind's chain); a natural number is what bears the ancestral of the successor relation to 0. Mathematical induction is a special case of ancestral induction. Now to avoid the need of infinite classes in the special case of natural number, by inversion as above, is all very well; but what of the general case? I answer this question by deriving the general from the special: defining the *n*th iterate of a relation *r*, for variable '*n*', with the help of number theory and then defining the ancestral of *r* in effect as the union of its iterates. Through this channel the general law of ancestral deduction is derived from what is usually looked upon as its special case, mathematical induction, and there is still no need of infinite classes. Incidentally, the notion of the iterate expedites the definition and treatment of arithmetical sum, product, and power. These developments are largely a revival of Dedekind's ideas.

Through these and subsequent developments an illusion is maintained of opulence unaccounted for by the axioms. The trick is the merging of the virtual theory with the real. The notation '$\{x: Fx\}$' of class abstraction is introduced by contextual definition in such a way that much use can be made of it even if the class does not exist; just its substitution for variables requires existence, and even this sanction is lightened somewhat by a use of schematic letters that does not require existence. We find we can enjoy a good deal of the benefit of a class without its existing either as a set or as an ultimate class.

After the natural numbers come the ratios and other real numbers. The reals are construed by essentially the Dedekind cut as usual, but the details of the development are so adjusted that the ratios turn out identical with the rational reals, not just isomorphic to them, and the reals become classes of natural numbers, not relations or classes of relations of them. The classical laws of real numbers, in particular that of the least bound, are found of course to depend on hypotheses of existence of infinite classes.

Then come the ordinal numbers, which I take in von Neumann's sense. My treatment of the natural numbers, earlier, is at variance with this course, for I take them rather in Zermelo's way. My reason is that I seem thus to be able to get by for a while with simpler existence axioms. In the general theory of ordinals we are bound to face serious existence assumptions, but we *can* preserve economy of assumption at the level of natural-number theory; and so I let the natural numbers and the ordinals go their separate ways.

I take comfort in the thought that it is just as well that students familiarize themselves with both Zermelo's and von Neumann's versions of the natural numbers. Still I would have been pleased to find that I could reap all advantages of the Zermelo version by adhering to von Neumann's from the start; and I appreciate the help of my student Kenneth Brown in exploring this alternative. To have rested with his best results, and so adhered first and last to von Neumann numbers, would have been nearly as good as the choice I made.

Transfinite recursion for me, as for von Neumann and Bernays, consists in specifying a transfinite sequence by specifying each thing in the sequence as a function of the preceding segment of the sequence. This matter is formalized in Chapter VIII, and put to use in defining the arithmetical operations on ordinals. It is put to use also in defining the *enumeration* of an arbitrary well-ordering. From the existence of enumerations, in turn, the comparability of well-orderings is deduced. These developments depend on existence assumptions which are written into the theorems as hypotheses. The same is true of the developments in the next two chapters, which are devoted to the Schröder-Bernstein theorem, the infinite cardinals, and the main equivalents of the axiom of choice.

The four concluding chapters are given over to the description and comparison of various systems of axiomatic set theory: Russell's theory of types, Zermelo's system, von Neumann's, two of mine, and, in a sketchy way, some recent developments. Logical connections are traced among them; for example, the theory of types is transformed essentially into Zermelo's system by translating it into general variables and taking the types as cumulative. The systems generally are given an unfamiliar cast by continuing to exploit the vestiges of the virtual theory of classes.

These four concluding chapters embody the origin of the book. One of my short lecture courses at Oxford, when I was there as George Eastman Visiting Professor in 1953–54, was a comparison of axiomatic set theories; and though I had dealt with the topic repeatedly at Harvard, it was the formulation at Oxford that got me to envisaging a little book. Then, for five years after Oxford, the project remained in abeyance while I finished a book on other matters. In 1959 I returned to this one, and that summer I gave a resumé of the material in some lectures at Tokyo. The book bade fair to be finished in a year, as a short book comprising the comparison of set theories along with some minimum preliminary chapters for orientation in the subject matter. But in the writing I got ideas that caused the preliminary chapters to run to five-sevenths of the book, and the book to take two additional years.

Since October the manuscript has had the inestimable benefit of critical readings by Professors Hao Wang, Burton S. Dreben, and John van Heijenoort. Their wise suggestions have led me to extend my coverage in some places, to deepen my analysis in others, to clear up some obscurities of exposition, to correct and supplement various of the historical notes, and, thanks to Wang particularly, to give a sounder interpretation of some papers. And all three readers have helped me in the asymptotic labor of spotting clerical errors.

I have been indebted also to Professors Dreben and John R. Myhill for helpful earlier remarks, and to current pupils for sundry details that will be attributed in footnotes. For defraying the costs of typing and other assistance I am grateful to the Harvard Foundation and the National Science Foundation (Grant GP-228).

W. V. Q.

Boston, January 1963

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INTRODUCTION

Set theory is the mathematics of classes. Sets are classes. The notion of class is so fundamental to thought that we cannot hope to define it in more fundamental terms. We can say that a class is any aggregate, any collection, any combination of objects of any sort; if this helps, well and good. But even this will be less help than hindrance unless we keep clearly in mind that the aggregating or collecting or combining here is to connote no actual displacement of the objects, and further that the aggregation or collection or combination of say seven given pairs of shoes is not to be identified with the aggregation or collection or combination of those fourteen shoes, nor with that of the twenty-eight soles and uppers. In short, a class may be thought of as an aggregate or collection or combination of objects just so long as 'aggregate' or 'collection' or 'combination' is understood strictly in the sense of 'class'.

We can be more articulate on the function of the notion of class. Imagine a sentence about something. Put a blank or variable where the *thing* is referred to. You have no longer a sentence about that particular thing, but an open sentence, so called, that may hold true of each of various things and false of others. Now the notion of class is such that there is supposed to be, in addition to the various things of which that sentence is true, also a further thing which is the *class* having each of those things and no others as member. It is the class *determined* by the open sentence.

Much the same characterization would serve to characterize the notion of attribute; for the notion of attribute is such that there is supposed to be, in addition to the various things of which a given open sentence is true, a further thing which is an *attribute* of each of those things and of no others. It, with apologies to McGuffey, is the *at*tribute that the open sentence at*trib*utes. But the difference, the only intelligible difference, between class and attribute emerges when to the above characterization of the notion of class we adjoin this needed supplement: classes are identical when their members are identical. This, the *law of extensionality*, is not considered to extend to attributes. If someone views attributes as identical always when they are attributes of the same things, he should be viewed as talking rather of classes. I deplore the notion of attribute, partly because of vagueness of the circumstances under which the attributes attributed by two open sentences may be identified.[1]

My characterization of the notion of class is not definitive. I was describing the function of the notion of class, not defining class. The description is incomplete in that a class is not meant to require, for its existence, that there be an open sentence to determine it. Of course, if we can specify the class at all, we can write an open sentence that determines it; the open sentence '$x \epsilon \alpha$' will do, where 'ϵ' means 'is a member of' and α is the class. But the catch is that there is in the notion of class no presumption that each class is specifiable. In fact there is an implicit presumption to the contrary, if we accept the classical body of theory that comes down from Cantor. For it is there proved that there can be no systematic way of assigning a different positive integer to every class of positive integers, whereas there is a systematic way (see §30) of assigning a different positive integer to every open sentence or other expression of any given language.

What my characterization of classes as determined by open sentences brings out is just the immediate utility and motivation of the notion of class, then, not its full range of reference. In fact the situation is yet worse: not only do classes outrun open

[1] In what I call referential opacity there is further cause for deploring the notion of attribute. On both complaints see *Word and Object*, pp. 209f.

sentences, but also conversely. An open sentence can be true of some things and false of others and yet fail, after all, to determine any class at all. Thus take the open sentence '$x \notin x$', which is true of an object x if and only if x is not a class that is a member of itself. If this open sentence determined a class y at all, we should have, for all x, $x \in y$ if and only if $x \notin x$; but then in particular $y \in y$ if and only if $y \notin y$, which is a contradiction. Such is Russell's paradox.[2] So, on the heels of finding that not all classes are determined by open sentences, we are now forced to recognize that not all open sentences determine classes. (These ills, by the way, would descend no less upon attributes.) A major concern in set theory is to decide, then, what open sentences to view as determining classes; or, if I may venture the realistic idiom, what classes there are. This is a question from which, in the course of this book, we shall never stray far.

The word 'set' recurred two sentences ago, for the first time since we took leave of it in the second sentence of this Introduction. It will be as well now to reckon a bit further with it. After all, it is on the cover.

Basically 'set' is simply a synonym of 'class' that happens to have more currency than 'class' in mathematical contexts. But this excess terminology is often used also to mark a technical distinction. As will emerge, there are advantages (and disadvantages) in holding with von Neumann and perhaps Cantor that not all classes are capable of being members of classes. In theories that hold this, the excess vocabulary has come in handy for marking the distinction; classes capable of being members are called sets. The others have lamely been called 'proper classes', on the analogy of 'Boston proper' or 'proper part'; I prefer to call them *ultimate* classes, in allusion to their not being members in turn of further classes.

We can know this technical sense of 'set' and still use the terms 'set' and 'class' almost interchangeably. For the distinction emerges only in systems that admit ultimate classes, and even in

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[2] Russell discovered it in 1901. He did not publish it until 1903, but meanwhile he discussed it in correspondence with Frege. The letters are to appear in van Heijenoort.

such systems the classes we have to do with tend to be sets rather than ultimate classes until we get pretty far out. And, as a name for the whole discipline, 'set theory' remains as defensible as 'class theory' even granted ultimate classes; for any properly general treatment of the sets would be bound anyway to relate them incidentally to the ultimate classes if such there be, thus covering the whole ground still. My own tendency will be to favor the word 'class' where 'class' or 'set' would do, except for calling the subject set theory. This is the usual phrase for the subject, and I should not like to seem to think that I was treating of something else.

In Chapter I we shall see how set theory can in part be simulated by purely notational convention, so that the appearance of talking of sets (or classes), and the utility of talking of them, are to some degree enjoyed without really talking of anything of the kind. This technique I call the "virtual theory of classes." When we move beyond it in later chapters to the real thing, we shall still retain this simulation technique as an auxiliary; for in a superficial way it will continue to offer some of the convenience of stronger existence assumptions that we actually make. But a difference between class and set is not to be sought in this, for the simulation is not a case of using things of some other sort to simulate sets; it is a case of seeming to talk of sets (or classes) when really talking neither of them nor of anything in their stead.

I defined set theory as the mathematical theory of classes, and went on to describe the notion of class. Yet I thereby gave no inkling of what prompts set theory. This is best done rather by quoting the opening sentence of Zermelo's paper of 1908: "Set theory is that branch of mathematics whose task is to investigate mathematically the fundamental notions of number, of order, and of function in their original simplicity, and to develop thereby the logical foundations of all arithmetic and analysis."[3]

Because of Russell's paradox and other antinomies, much of set theory has to be pursued more self-consciously than many

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[3] The English is Bauer-Mengelberg's, from van Heijenoort's *Source Book.*

other parts of mathematics. The natural attitude on the question what classes exist is that any open sentence determines a class. Since this is discredited, we have to be deliberate about our axioms of class existence and explicit about our reasoning from them; intuition is not in general to be trusted here. Moreover, since the known axiom systems for the purpose present a variety of interesting alternatives, none of them conclusive, it would be imprudent at the present day to immerse ourselves in just one system to the point of retraining our intuition to it. The result is that the logical machinery is more in evidence in this part of mathematics than in most.

But in this respect the literature on set theory divides conspicuously into two parts. The part that concerns itself mainly with foundations of analysis gets on with much the same measure of informality as other parts of mathematics. Here the sets concerned are primarily sets of real numbers, or of points, and sets of such sets and so on. Here the antinomies do not threaten, for questions like '$x \in x$' do not come up.

It is rather in what Fraenkel has called abstract set theory, as against point-set theory, that we have so particularly to call our shots. A book in this branch typically takes up, in order, the following topics. First there are the general assumptions of the existence of classes, and other general laws concerning them. Then there is the derivation of a theory of relations from this basis, and more particularly a theory of functions. Then the integers are defined, and the ratios, and the real numbers, and the arithmetical laws are derived that govern them. Finally one gets on to infinite numbers: the theory of the relative sizes of infinite classes and the relative lengths of infinite orderings. These latter matters are the business of set theory at its most characteristic. They are the discovery or creation of Cantor, and thus virtually coeval with set theory itself.

This is a book on abstract set theory, and it falls generally into the above outline. It thus belongs to the branch of set-theoretic literature that has to be rather explicit about its logic. And this requirement is somewhat heightened, in this book, by two special circumstances. One is that in the last four chapters I shall com-