CLASSICAL DYNAMICS

R. H. ATKIN

CLASSICAL DYNAMICS

by

R. H. ATKIN, M.A., M.Sc.

Principal of Unischol Tutorial College, London
Formerly Senior Lecturer in Mathematics,
Northampton College of Advanced Technology, London

HEINEMANN

LONDON MELBOURNE TORONTO

PREFACE

I HAVE written this book in the belief that it is possible to learn Dynamics and answer examination questions at the same time. Indeed, the number of such questions which have accumulated over the last century and which have circulated in the textbooks and examination papers is so great that the weak student may perhaps be forgiven for suspecting that the subject consists of a bewildering set of tricks especially devised for their solution.

In fact, the foundations of the subject consist only of a few simple experimental laws and physical observations, but they are applicable to a very wide selection of the natural phenomena of everyday experience. It therefore seems sensible to present the subject by first explaining the ideas which are involved, in a suitable algebraic framework, and then illustrating the scope of their applications by solving a large number of typical problems. By selecting most of these problems from university examination papers I have tried to take some of the sting out of these as well.

The theory of vectors has recently provided a novel approach to many dynamical problems, particularly those concerned with non-holonomic systems, and it is now desirable to weave the vector methods and the analytical methods into some sort of harmony. The final emphasis will really be a matter of individual taste, but the student would be ill-advised to condemn one or other of the techniques until he has thoroughly appreciated its possibilities. To acquire facility in using vector methods it is important to understand the concepts of " linearly independent vectors " and " rotating frames of reference " and also to be able to expand certain vector products. Because of this, and to ensure that all the relevant theory is available, I have included a chapter on Vector Algebra ; this, in spite of the fact that many textbooks now have chapters on " vectors," although the effect on our students is as yet far from decisive.

I have been greatly helped by the kindness of Professor H. Bondi and of Dr. L. Pincherle, both of whom read the manuscript, and I was glad to be able to take advantage of much of their advice.

My thanks are due both to the University of London for permission to reprint examination questions and to the Syndics of the Cambridge University Press for permission to reproduce questions set in the Tripos papers and also questions which are to be found in *Theoretical Mechanics*, by A. E. H. Love. These latter questions are denoted by the letter (C.) in the exercises at the ends of the chapters.

Finally, it does not seem inappropriate to commend the publishers once more for their persistence and daring in encouraging me to write the book.

R. H. ATKIN

London, 1959.

CONTENTS

CHAPTER I

HISTORICAL FOUNDATIONS

1.1. Galileo Galilei *

IT is truly said that the modern science of dynamics was founded by Galileo.

He was the first man to systematically investigate the motion of falling bodies. Prior to this men had generally felt reasonably satisfied with the vague Aristotelian view that every object must seek its "place" in the universe: thus heavy bodies must fall and light ones rise or, at least, fall less quickly. This was the *why* of falling bodies. Galileo set himself the task of discovering *how* bodies fall, and in succeeding admirably in this he was in fact laying the intellectual foundations of what we now call science.

By careful experiments with a ball rolling down an inclined plane Galileo discovered the formulæ $v = gt$ and $s = \frac{1}{2}gt^2$, where g is a constant, and here it is as well to remember that there were no clocks available which he could use for accurate measurements of small intervals of time. The constant g, being the ratio of velocity to time, is a measure of *acceleration*, and this idea too was contributed by Galileo. Thus he had solved the general problem of the motion of a body which moves in a straight line with a constant acceleration.

Now the fact that a heavy body falls to the ground had always been associated with the fact that a heavy body has a *weight*, i.e. is acted on by a *force*, but, before Galileo, it was not known in what sense force produces motion. It was Galileo who first perceived that force (or pressure) is not determinative of either position or velocity but of acceleration, or change of velocity. And here it is important for us to appreciate the concept of force in its historical context.

The word *force*, or forces, had arisen as an equivalence to the words *circumstances which determine motion*, and once the precise nature of this determination had been discovered (that is to say, "that which determines motion" is "that which determines acceleration") it was immediately possible to deduce the Law of Inertia, viz. "when there are no forces acting on a body it moves in a straight line with constant (including zero) velocity."

In connection with his work on bodies falling under gravity (or under their weights) Galileo was also able to solve the problem of a projectile with oblique projection. His method was that found in our

* Italian mathematician-scientist. Born in Pisa 1564. Died 1642.

early school text-books, but we mention it here because it illustrates the idea that separate forces acting on a body produce their own accelerations which, in the proper circumstances, can be separately treated in finding velocities and distances. Thus, in the case of the projectile we have a constant acceleration in the direction of the downward vertical and a zero acceleration horizontally. The distances travelled in a time t in these two directions, assuming the velocity of projection is V inclined at an angle α to the horizontal and below the horizontal, will be simply

$$y = Vt \sin \alpha + \tfrac{1}{2}gt^2 \quad \text{(downwards)}$$

and

$$x = Vt \cos \alpha \quad \text{(horizontally)}$$

These immediately give the *parabolic arc*

$$y = x \tan \alpha + \frac{gx^2}{2V^2 \cos^2 \alpha}$$

The fact that these two motions take place simultaneously and independently illustrates the principle of the *parallelogram law* for accelerations. This parallelogram rule was already an accepted part of Statics, having been ably demonstrated by the Dutch mathematician Simon Stevinus.* Generally speaking we can say that it is the *independence* of the forces (accelerations) which results in a parallelogram law for their composition. This property of independence enables us to identify a one-to-one correspondence between the "vector" and a geometrical displacement, and two simultaneous displacements are equivalent to a third which lies along the relevant diagonal of a certain parallelogram. These ideas we shall analyse more fully when we come to consider vectors in Chapter II.

Finally we shall mention Galileo's examination of the oscillations of a simple pendulum, both the isochronism and the general form of the formula for the periodic time being known to him. Furthermore, motion on a curve (the circular arc described by the bob of the pendulum) meant for Galileo motion on a succession of infinitesimal inclined planes, and with the aid of the pendulum he was able to show that the velocity acquired by a heavy body in moving freely down an inclined plane was dependent only on the *height* through which it had effectively fallen. Thus the bob of a pendulum will rise on one side of the lowest position just as high as it has previously fallen on the other, and by suddenly shortening the length of string when the bob was in the lowest position (by placing stops at appropriate points) this property was shown to be independent of the radius of the arc described—and therefore independent of the particular "succession of inclined planes" described.

* 1548–1620. Stevinus investigated equilibrium on an inclined plane; demonstrated the resolution of forces as well as their composition, and distinguished between stable and unstable equilibrium.

It is clear that this result amounted to the principle of the *conservation of energy*, a principle which we now know to be of tremendous significance.

1.2 Christiaan Huygens *

In his contribution to dynamics Huygens proved, among other things, that a body which describes a circle of radius r with a constant velocity v must experience an acceleration towards the centre of $\frac{v^2}{r}$. This involved the idea that velocity possesses not only *magnitude* but also *direction*, and that acceleration can mean change of direction of velocity even when there is no change in the magnitude of the velocity.

Huygens also did a great deal of work on the pendulum and invented the first pendulum clock—inventing the escapement on the way. In this connection he went much further than did Galileo in that he solved the problem of the "centre of oscillation" of a compound pendulum. This, of course, involved the dynamics of several connected bodies (or particles of a rigid body) and his method involved such an original concept that it is worth discussing in more detail.

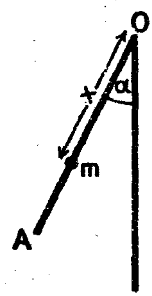

FIG. 1.

Consider a rigid body in the form of a heavy rod OA which is free to move in a vertical plane about a horizontal axis through one end O, as in Fig. 1. Now regard the rod as made up of a large number of small masses $m_1, m_2, \ldots$ at distances $x_1, x_2, \ldots$ from the end O, and let m in the diagram be a typical mass. Suppose now that we let the rod swing freely under gravity from its extreme position in which it is inclined at an angle α to the downward vertical. Then Huygens argued that *the centre of gravity of the rod must rise through the same distance as it falls.* This is an imaginative extension of Galileo's result obtained for the motion of a single mass, and again we can clearly see that it amounts to the modern principle of the conservation of energy.

Suppose the centre of gravity falls a distance z by the time the rod reaches the vertical position, then

$$z \sum m = m_1 x_1 \cos\alpha + m_2 x_2 \cos\alpha + \ldots$$
$$= \cos\alpha \sum mx$$

giving

$$z = \cos\alpha \frac{\sum mx}{\sum m} \qquad \ldots \ldots \quad (1)$$

* 1629–95. Dutch mathematician, famous also for his wave theory of light and Huygens' Principle.

Let the point P on OA, where $OP = 1$, acquire a velocity u during the downward swing, then the mass m acquires a velocity xu. But Galileo showed that a body with velocity v will rise a height $\frac{v^2}{2g}$ and so each particle of the rod will rise a height $\frac{u^2}{2g} \times x^2$. The centre of gravity will therefore rise a height z' given by

$$z' \sum m = \frac{u^2}{2g} \sum mx^2 \quad . \quad . \quad . \quad (2)$$

If we put $z' = z$, (1) and (2) give

$$\frac{u^2}{2g} \sum mx^2 = \cos a \sum mx \quad . \quad . \quad . \quad (3)$$

To find the length of the equivalent simple pendulum (say l) we can apply (3) to the single mass of the imagined bob whence

$$\frac{u^2}{2g} . l = \cos a \quad . \quad . \quad . \quad . \quad (4)$$

Then (3) and (4) give $$l = \frac{\sum mx^2}{\sum mx}$$

Writing G as the centre of gravity and $OG = \bar{x}$, $\sum m = M$ we get

$$l = \frac{\sum mx^2}{M\bar{x}} \quad . \quad . \quad . \quad . \quad (5)$$

We thus see that Huygens had also discovered the quantity $\sum mx^2$, now known (after Euler) as the *moment of inertia* of the rod about the axis through O. Indeed he went on to obtain the "parallel axes" theorem and the complete analysis of what we now call the compound pendulum.

Other contributions by Huygens included an analysis of the cycloid and its isochronous property and furthermore, in 1669, he submitted to the Royal Society his work on the impact of elastic bodies. In this latter problem he obtained the correct solution, indicating therein the significance of the product mass $\times$ velocity (momentum). Here we must mention that both Wallis and Wren had submitted equally correct solutions at the end of 1668, the one dealing with collisions of inelastic bodies and the other with elastic collisions. The final form of the laws of impact was to be provided by Newton and published in his *Principia*.

1.3 Isaac Newton *

From the point of view of this study the two great achievements of Newton were (*a*) his theory of universal gravitation, and (*b*) his formulation of the principles of mechanics. It is very important for us to examine, in some detail, the concepts which he introduced and clarified in the course of his work.

Ever since Galileo had invented his telescope men had been studying the motions of the planets with ever increasing interest and accuracy. In particular, a great deal of observed data had been collected by Tycho Brahe,† and from this Kepler ‡ had deduced his famous three laws describing the motion of the planets about the sun. These amounted to :

(1) The planets describe ellipses with the sun situated at a focus.
(2) The radius vector joining the sun with a planet describes equal areas in equal times, i.e. the rate of description of sectorial area is constant.
(3) The cubes of the mean distances of the planets from the sun are proportional to the squares of their times of revolution, i.e. if $2a$ is the major axis of the elliptic orbit and t is the periodic time then $t^2 \propto a^3$.

Newton was able to show that these laws were compatible with the assumption that each planet possesses an acceleration towards the sun which is inversely proportional to the square of their distance from it. Furthermore he saw this acceleration as being of the same nature as that experienced by bodies falling near the earth's surface. This remarkable generalisation led him to the concept that all bodies, taken in pairs, induce in each other mutual accelerations. Translating this into terms of *force* requires a new principle and Newton supplied this in his law of "action and reaction"—and this in its turn provides us with a view of *mass* not possessed by any of Newton's predecessors, a concept which distinguishes between mass and heaviness (or weight).

The *laws of motion* which Newton published in his *Principia* amount to the following :

Law I Every body perseveres in its state of rest or of uniform motion in a straight line except in so far as it is compelled to change that state by impressed forces.

Law II Change of momentum is proportional to the impressed force and takes place along the line of action of that force.

* 1642–1727. English mathematician. His great work, *Philosophiae Naturalis Principia Mathematica*, was published in 1687.

† Danish astronomer, 1546–1601.

‡ German mathematician, 1571–1630.

Law III Action and reaction are always equal and opposite; that is to say, the actions of two bodies upon each other are equal and directly opposite.

Newton perceived that a body possesses an invariable property known as its mass and that, when it possesses an acceleration f then the force acting on the body will be $P = kmf$, where k is a constant of proportionality. In modern notation Law II will be written Force $\propto \frac{d}{dt}(mv)$ or $m\frac{dv}{dt}$ when the mass m does not artificially change with time. Thus we see that the weight of a body, being the force mg, can vary if g varies, whereas the mass m will at the same time remain constant. Furthermore, the *masses* of two bodies can be accurately *compared* by weighing them in the two pans of a balance and the *weight* of any one body will be found by weighing it with a spring balance. If we suppose the units of measurement to be suitably chosen we can write $k = 1$ and Law II as

$$\text{Force} = \text{mass} \times \text{acceleration}$$

Then Law III, applied to two bodies A and B of masses m_1 and m_2 respectively (Fig. 2), says that the *mutual forces* P and Q are equal.

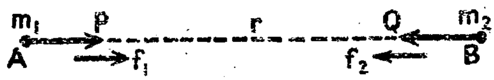

FIG. 2.

If A possesses acceleration f_1 and B f_2 we get

$$P = m_1 f_1 = m_2 f_2 = Q$$

and so
$$\frac{m_1}{m_2} = \frac{f_2}{f_1} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Unfortunately Newton defined mass as density × volume, and since density involves the idea of mass this "definition" is clearly inadequate. Since Law II attempts to relate mass to force it is necessary to have a logically independent view of the one before we can talk about the other. Law III also involves the idea of mass. Considering also the rather obvious fact that Law I is an immediate deduction from Law II it is clear that these laws are not the most economical possible.

A set of propositions designed to reduce Newton's laws to their simplest and most economical in thought was given by E. Mach at the beginning of this century. These emphasise the experimental nature of the foundations of mechanics and are as follows:

I. Experimental proposition. Two bodies set opposite each other induce in each other opposite accelerations in the direction of their line of junction.

Definition. The mass-ratio of any two bodies is the numerical value of the inverse ratio of their mutually induced accelerations.

Definition. Moving force is the product mass × acceleration for any body.

II. Experimental proposition. The accelerations which any number of bodies A_1, A_2, . . . induce in a body B, are independent of each other.

The two definitions of mass and force lead deductively to Newton's law of action–reaction and, of course, include Laws II and I. The second experimental proposition implies the *parallelogram of forces*, which was itself explicitly stated by Newton in a corollary to his three laws of motion.

Units of mass, force

The units of mass are the *pound* (lb.) and the *gramme* (gm.).

The units of force are the *poundal* (when mass is in pounds and acceleration in ft./sec./sec.), and the *dyne* (when mass is in grammes and acceleration in cm./sec./sec.). The two systems are referred to as the ft. lb. sec.-system and the c.g.s.-system.

When forces are measured in terms of weights, as they can be, this is indicated by calling them pounds-weight (lb. wt.) or grammes-weight (gm. wt.). The law of motion being now

$$\text{Force (dynes)} = \text{mass (gm.)} \times \text{acceleration (c.g.s.)}$$

we have

$$1 \text{ gm. wt.} = 1 \text{ gm.} \times g \text{ cm./sec./sec.}$$
$$= g \text{ dynes}$$

Thus ∴ x gm. wt. $= gx$ dynes

and x lb. wt. $= gx$ poundals

The values of g in the two systems of units are 32·2 ft./sec./sec. and 981 cm./sec./sec. The basic (Newtonian) equation, viz. $F = \frac{d}{dt}(mv)$, holds only in terms of these *absolute units*.

1.4 Universal gravitation

As mentioned in 1.3, Newton solved the age-old problem of the motion of the planets about the sun by ascribing to each planet an acceleration towards the sun proportional to $\frac{1}{r^2}$, r being the distance between sun and planet. In terms of force we can now write :

> *every body attracts every other body with a force which is inversely proportional to the square of their distance apart and directly proportional to the product of their masses.*

Thus, in Fig. 2 of 1.3, we should write

$$P = Q = \gamma\frac{m_1 m_2}{r^2} \quad . \quad . \quad . \quad . \quad (1)$$

where γ is a universal constant.

If now, for example, A is the sun and B is a planet, the acceleration of B towards A will be

$$f_2 = \frac{Q}{m^2} = \gamma\frac{m_1}{r^2} = \frac{\mu}{r^2} \quad . \quad . \quad . \quad . \quad (2)$$

where μ will be the same for all planets of the sun. The value of γ has been determined to be $6{\cdot}66 \times 10^{-8}$ c.g.s. units.

It is also clear from (1) that ideally the distance r can only be known precisely when the masses at A and B are identified as very small *particles* of matter, occupying negligible volumes. In dealing with finite distributions of matter we are therefore obliged to use the methods of the integral calculus involving, as they do, limiting processes in which the objects of calculation are divided into infinitesimal elements. There are one or two simple cases which, by way of illustration, we can very well examine at this stage.

Example 1. To find the gravitational attraction on a particle of unit mass due to a uniform rod AB of density ρ per unit length.

Let the point P be a distance p from the line of the rod, which we

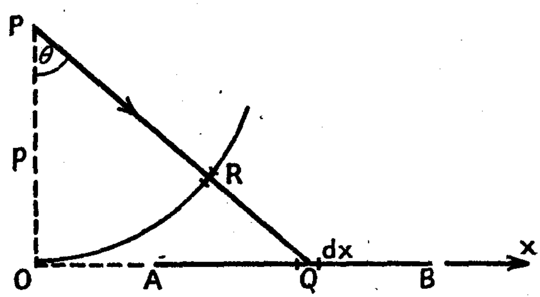

FIG. 3.

take to be the x-axis in Fig. 3. Let $\rho\, dx$ be an element of mass at any point Q of AB, then the attraction at P of this will be $\gamma . \frac{\rho\, dx}{PQ^2}$. Writing $OQ = x = p \tan\theta$ this becomes

$$\gamma\rho . \frac{p^2 \sec^2\theta}{p \sec^2\theta}\, d\theta = \gamma\frac{\rho p\, d\theta}{p^2}$$

which equals the attraction at P of the corresponding arc at R of the circle, centre P, radius p. Thus the resultant attraction of the rod AB at P bisects the angle APB (see Fig. 4) and is of magnitude

$$F = 2\int_0^\alpha \frac{\gamma\rho}{p}.\cos\phi\, d\phi = \frac{2\gamma\rho}{p}\sin\alpha = \frac{2\gamma\rho}{p}\sin(\tfrac{1}{2}\angle APB)$$

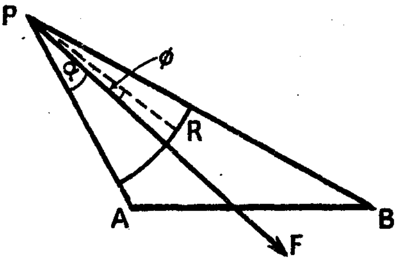

FIG. 4.

Example 2. To find the attraction per unit mass at a point P which is either outside or inside a uniform spherical shell of density ρ and radius a.

Let P be outside the shell as shown in Fig. 5 with $OP = c > a$.

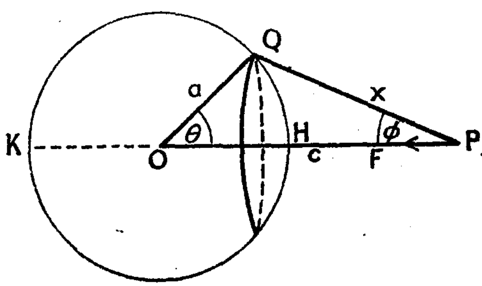

FIG. 5.

Divide the surface into small circles around OP as axis as shown. Then each point on such a small circle is the same distance x from P and by symmetry the resultant attraction F at P is along PO. The attraction at P due to any one small circle will be

$$\gamma\frac{(\text{mass})}{x^2}\cos\phi = \gamma\frac{\rho\,.\,2\pi a^2\sin\theta\cos\phi}{x^2}.d\theta$$

Since $x^2 = a^2 + c^2 - 2ac\cos\theta$ it follows that $x\,dx = ac\sin\theta\, d\theta$ and we get

$$F = 2\pi\rho\gamma\int_{PH}^{PK}\frac{ax\cos\phi}{cx^2}\,dx$$

Since $a^2 = x^2 + c^2 - 2xc\cos\phi$ this becomes

$$F = \frac{\pi\rho\gamma.a}{c^2}\int_{PH}^{PK}\left\{1 + \frac{c^2 - a^2}{x^2}\right\}dx$$

When P is outside the shell $PH = c - a$, $PK = c + a$.
When P is inside the shell $PH = a - c$, $PK = a + c$.

Thus when P lies outside the shell $F = \gamma.\frac{4\pi a^2\rho}{c^2} = \gamma\frac{M}{c^2}$, where M = total mass, and in this case *the shell behaves like a particle of mass M at the centre O.*

When P lies inside the shell we get $F = 0$ (all c).

Corollary. We can now find the attraction at a point P due to a uniform solid sphere of radius a.

(i) When P lies outside the sphere it is clear that the attraction is

$$\gamma\frac{(\text{Mass})}{c^2} = \tfrac{4}{3}.\gamma\pi\rho a^3/c^2$$

(ii) When P lies inside we get, from Fig. 6,

$$\text{Attraction} = \gamma(\tfrac{4}{3}\pi\rho c^3)/c^2 = \tfrac{4}{3}\pi\gamma\rho c$$

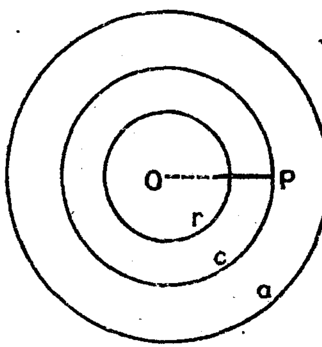

FIG. 6.

1.5 Laws of impact

We have already noticed that the problems of the collisions between perfectly elastic bodies (those which completely recover their former shapes after deformation) and between inelastic bodies (those which suffer permanent deformation) had been studied and solved by Wren, Huygens and Wallis. All three had found that the separate *momenta* of the colliding bodies suffered sudden changes, i.e. suffered *finite discontinuities.*

Newton also conducted elaborate experiments on the problem of impact and published his laws relating to it in *Principia.* If two rolling smooth spheres (e.g.) meet in collision on a horizontal table the momentum of each changes suddenly through a finite amount. We say that each sphere has received an *impulsive blow* or *impulse*, and that this impulse equals the change in the momentum. Thus (v. Fig. 7) if the velocities of A and B are u_1, v_1 just before impact and u_2, v_2 just after impact, if the the mass of A is M and of B is M' we shall have

$$Mu_2 - Mu_1 = -I \quad . \quad . \quad . \quad . \quad (1)$$

$$M'v_2 - M'v_1 = I \quad . \quad . \quad . \quad . \quad (2)$$

Notice that here again the law of action–reaction is invoked to say that the impulses on A and B are equal and opposite. This can be

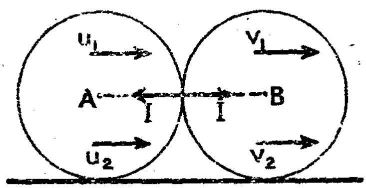

FIG. 7.

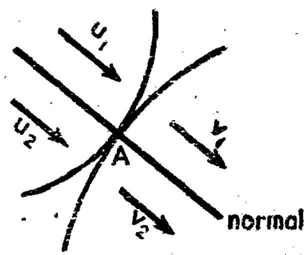

FIG. 8.

directly confirmed experimentally since, by adding (1) and (2), we get

$$Mu_1 + M'v_1 = Mu_2 + M'v_2 \quad . \quad . \quad . \quad (3)$$

The total momentum remains unchanged.

The question to be settled is now, " in what proportion is the initial momentum shared between the two bodies immediately after impact ? " *Newton's experimental law* provides the answer in saying

$$\textit{Velocity of separation} = e \times \textit{Velocity of approach} \quad . \quad (4)$$

where e is a constant, known as the *coefficient of restitution*, and which depends on the elastic properties of the two bodies. Thus (4) means we can write

$$v_2 - u_2 = e(u_1 - v_1) \quad . \quad . \quad . \quad (5)$$

whence (3) and (5) solve the problem.

We shall now take a generalisation of (4) as a basis for solving more diverse problems of collision. We shall say that (v. Fig. 8) *whenever two smooth surfaces collide the velocities of the respective bodies along the line of the common normal* (at A) *are related before and after impact by equation* (4).

1.6 Conclusion

In considering the dynamics of a single body the principles of Galilso-Newton lead us to the following point of view.