Handbook of Mathematical Functions

With

Formulas, Graphs, and Mathematical Tables

Edited by

Milton Abramowitz and Irene A. Stegun

UNITED STATES DEPARTMENT OF COMMERCE • Luther H. Hodges, *Secretary*
NATIONAL BUREAU OF STANDARDS • A. V. Astin, *Director*

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Preface

The present volume is an outgrowth of a Conference on Mathematical Tables held at Cambridge, Mass., on September 15–16, 1954, under the auspices of the National Science Foundation and the Massachusetts Institute of Technology. The purpose of the meeting was to evaluate the need for mathematical tables in the light of the availability of large scale computing machines. It was the consensus of opinion that in spite of the increasing use of the new machines the basic need for tables would continue to exist.

Numerical tables of mathematical functions are in continual demand by scientists and engineers. A greater variety of functions and higher accuracy of tabulation are now required as a result of scientific advances and, especially, of the increasing use of automatic computers. In the latter connection, the tables serve mainly for preliminary surveys of problems before programming for machine operation. For those without easy access to machines, such tables are, of course, indispensable.

Consequently, the Conference recognized that there was a pressing need for a modernized version of the classical tables of functions of Jahnke-Emde. To implement the project, the National Science Foundation requested the National Bureau of Standards to prepare such a volume and established an Ad Hoc Advisory Committee, with Professor Philip M. Morse of the Massachusetts Institute of Technology as chairman, to advise the staff of the National Bureau of Standards during the course of its preparation. In addition to the Chairman, the Committee consisted of A. Erdélyi, M. C. Gray, N. Metropolis, J. B. Rosser, H. C. Thacher, Jr., John Todd, C. B. Tompkins, and J. W. Tukey.

The primary aim has been to include a maximum of useful information within the limits of a moderately large volume, with particular attention to the needs of scientists in all fields. An attempt has been made to cover the entire field of special functions. To carry out the goal set forth by the Ad Hoc Committee, it has been necessary to supplement the tables by including the mathematical properties that are important in computation work, as well as by providing numerical methods which demonstrate the use and extension of the tables.

The Handbook was prepared under the direction of the late Milton Abramowitz, and Irene A. Stegun. Its success has depended greatly upon the cooperation of many mathematicians. Their efforts together with the cooperation of the Ad Hoc Committee are greatly appreciated. The particular contributions of these and other individuals are acknowledged at appropriate places in the text. The sponsorship of the National Science Foundation for the preparation of the material is gratefully recognized.

It is hoped that this volume will not only meet the needs of all table users but will in many cases acquaint its users with new functions.

ALLEN V. ASTIN, *Director.*

Washington, D.C.

Foreword

This volume is the result of the cooperative effort of many persons and a number of organizations. The National Bureau of Standards has long been turning out mathematical tables and has had under consideration, for at least 10 years, the production of a compendium like the present one. During a Conference on Tables, called by the NBS Applied Mathematics Division on May 15, 1952, Dr. Abramowitz of that Division mentioned preliminary plans for such an undertaking, but indicated the need for technical advice and financial support.

The Mathematics Division of the National Research Council has also had an active interest in tables; since 1943 it has published the quarterly journal, "Mathematical Tables and Aids to Computation" (MTAC), editorial supervision being exercised by a Committee of the Division.

Subsequent to the NBS Conference on Tables in 1952 the attention of the National Science Foundation was drawn to the desirability of financing activity in table production. With its support a 2-day Conference on Tables was called at the Massachusetts Institute of Technology on September 15–16, 1954, to discuss the needs for tables of various kinds. Twenty-eight persons attended, representing scientists and engineers using tables as well as table producers. This conference reached consensus on several conclusions and recommendations, which were set forth in the published Report of the Conference. There was general agreement, for example, "that the advent of high-speed computing equipment changed the task of table making but definitely did not remove the need for tables". It was also agreed that "an outstanding need is for a Handbook of Tables for the Occasional Computer, with tables of usually encountered functions and a set of formulas and tables for interpolation and other techniques useful to the occasional computer". The Report suggested that the NBS undertake the production of such a Handbook and that the NSF contribute financial assistance. The Conference elected, from its participants, the following Committee: P. M. Morse (Chairman), M. Abramowitz, J. H. Curtiss, R. W. Hamming, D. H. Lehmer, C. B. Tompkins, J. W. Tukey, to help implement these and other recommendations.

The Bureau of Standards undertook to produce the recommended tables and the National Science Foundation made funds available. To provide technical guidance to the Mathematics Division of the Bureau, which carried out the work, and to provide the NSF with independent judgments on grants for the work, the Conference Committee was reconstituted as the Committee on Revision of Mathematical Tables of the Mathematics Division of the National Research Council. This, after some changes of membership, became the Committee which is signing this Foreword. The present volume is evidence that Conferences can sometimes reach conclusions and that their recommendations sometimes get acted on.

Active work was started at the Bureau in 1956. The overall plan, the selection of authors for the various chapters, and the enthusiasm required to begin the task were contributions of Dr. Abramowitz. Since his untimely death, the effort has continued under the general direction of Irene A. Stegun. The workers at the Bureau and the members of the Committee have had many discussions about content, style and layout. Though many details have had to be argued out as they came up, the basic specifications of the volume have remained the same as were outlined by the Massachusetts Institute of Technology Conference of 1954.

The Committee wishes here to register its commendation of the magnitude and quality of the task carried out by the staff of the NBS Computing Section and their expert collaborators in planning, collecting and editing these Tables, and its appreciation of the willingness with which its various suggestions were incorporated into the plans. We hope this resulting volume will be judged by its users to be a worthy memorial to the vision and industry of its chief architect, Milton Abramowitz. We regret he did not live to see its publication.

P. M. Morse, *Chairman*.
A. Erdélyi
M. C. Gray
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1. Introduction

The present Handbook has been designed to provide scientific investigators with a comprehensive and self-contained summary of the mathematical functions that arise in physical and engineering problems. The well-known Tables of Functions by E. Jahnke and F. Emde has been invaluable to workers in these fields in its many editions[1] during the past half-century. The present volume extends the work of these authors by giving more extensive and more accurate numerical tables, and by giving larger collections of mathematical properties of the tabulated functions. The number of functions covered has also been increased.

The classification of functions and organization of the chapters in this Handbook is similar to that of An Index of Mathematical Tables by A. Fletcher, J. C. P. Miller, and L. Rosenhead.[2] In general, the chapters contain numerical tables, graphs, polynomial or rational approximations for automatic computers, and statements of the principal mathematical properties of the tabulated functions, particularly those of computational importance. Many numerical examples are given to illustrate the use of the tables and also the computation of function values which lie outside their range. At the end of the text in each chapter there is a short bibliography giving books and papers in which proofs of the mathematical properties stated in the chapter may be found. Also listed in the bibliographies are the more important numerical tables. Comprehensive lists of tables are given in the Index mentioned above, and current information on new tables is to be found in the National Research Council quarterly Mathematics of Computation (formerly Mathematical Tables and Other Aids to Computation).

The mathematical notations used in this Handbook are those commonly adopted in standard texts, particularly Higher Transcendental Functions, Volumes 1–3, by A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi (McGraw-Hill, 1953–55). Some alternative notations have also been listed. The introduction of new symbols has been kept to a minimum, and an effort has been made to avoid the use of conflicting notation.

2. Accuracy of the Tables

The number of significant figures given in each table has depended to some extent on the number available in existing tabulations. There has been no attempt to make it uniform throughout the Handbook, which would have been a costly and laborious undertaking. In most tables at least five significant figures have been provided, and the tabular intervals have generally been chosen to ensure that linear interpolation will yield four- or five-figure accuracy, which suffices in most physical applications. Users requiring higher precision in their interpolates may obtain them by use of higher-order interpolation procedures, described below.

In certain tables many-figured function values are given at irregular intervals in the argument. An example is provided by Table 9.4. The purpose of these tables is to furnish "key values" for the checking of programs for automatic computers; no question of interpolation arises.

The maximum end-figure error, or "tolerance" in the tables in this Handbook is $\frac{6}{10}$ of 1 unit everywhere in the case of the elementary functions, and 1 unit in the case of the higher functions except in a few cases where it has been permitted to rise to 2 units.

[1] The most recent, the sixth, with F. Loesch added as co-author, was published in 1960 by McGraw-Hill, U.S.A., and Teubner, Germany.

[2] The second edition, with L. J. Comrie added as co-author, was published in two volumes in 1962 by Addison-Wesley, U.S.A., and Scientific Computing Service Ltd., Great Britain.

3. Auxiliary Functions and Arguments

One of the objects of this Handbook is to provide tables or computing methods which enable the user to evaluate the tabulated functions over complete ranges of real values of their parameters. In order to achieve this object, frequent use has been made of auxiliary functions to remove the infinite part of the original functions at their singularities, and auxiliary arguments to cope with infinite ranges. An example will make the procedure clear.

The exponential integral of positive argument is given by

$$\mathrm{Ei}(x)=\int_{-\infty}^{x}\frac{e^u}{u}du$$

$$=\gamma+\ln x+\frac{x}{1\cdot 1!}+\frac{x^2}{2\cdot 2!}+\frac{x^3}{3\cdot 3!}+\ldots$$

$$\sim\frac{e^x}{x}\left[1+\frac{1!}{x}+\frac{2!}{x^2}+\frac{3!}{x^3}+\ldots\right]_{(x\to\infty)}$$

The logarithmic singularity precludes direct interpolation near $x=0$. The functions $\mathrm{Ei}(x)-\ln x$ and $x^{-1}[\mathrm{Ei}(x)-\ln x-\gamma]$, however, are well-behaved and readily interpolable in this region. Either will do as an auxiliary function; the latter was in fact selected as it yields slightly higher accuracy when $\mathrm{Ei}(x)$ is recovered. The function $x^{-1}[\mathrm{Ei}(x)-\ln x-\gamma]$ has been tabulated to nine decimals for the range $0\leq x\leq\frac{1}{2}$. For $\frac{1}{2}\leq x\leq 2$, $\mathrm{Ei}(x)$ is sufficiently well-behaved to admit direct tabulation, but for larger values of x, its exponential character predominates. A smoother and more readily interpolable function for large x is $xe^{-x}\mathrm{Ei}(x)$; this has been tabulated for $2\leq x\leq 10$. Finally, the range $10\leq x\leq\infty$ is covered by use of the inverse argument x^{-1}. Twenty-one entries of $xe^{-x}\mathrm{Ei}(x)$, corresponding to $x^{-1}=.1(-.005)0$, suffice to produce an interpolable table.

4. Interpolation

The tables in this Handbook are not provided with differences or other aids to interpolation, because it was felt that the space they require could be better employed by the tabulation of additional functions. Admittedly aids could have been given without consuming extra space by increasing the intervals of tabulation, but this would have conflicted with the requirement that linear interpolation is accurate to four or five figures.

For applications in which linear interpolation is insufficiently accurate it is intended that Lagrange's formula or Aitken's method of iterative linear interpolation[3] be used. To help the user, there is a statement at the foot of most tables of the maximum error in a linear interpolate, and the number of function values needed in Lagrange's formula or Aitken's method to interpolate to full tabular accuracy.

As an example, consider the following extract from Table 5.1.

x	$xe^xE_1(x)$	x	$xe^xE_1(x)$
7.5	.89268 7854	8.0	.89823 7113
7.6	.89384 6312	8.1	.89927 7888
7.7	.89497 9666	8.2	.90029 7306
7.8	.89608 8737	8.3	.90129 6073
7.9	.89717 4302	8.4	.90227 4695

$$\begin{bmatrix}(-6)3\\ 5\end{bmatrix}$$

The numbers in the square brackets mean that the maximum error in a linear interpolate is 3×10^{-6}, and that to interpolate to the full tabular accuracy five points must be used in Lagrange's and Aitken's methods.

[3] A. C. Aitken, On interpolation by iteration of proportional parts, without the use of differences, Proc. Edinburgh Math. Soc. **3**, 56–76 (1932).

Let us suppose that we wish to compute the value of $xe^xE_1(x)$ for $x=7.9527$ from this table. We describe in turn the application of the methods of linear interpolation, Lagrange and Aitken, and of alternative methods based on differences and Taylor's series.

(1) Linear interpolation. The formula for this process is given by

$$f_p=(1-p)f_0+pf_1$$

where f_0, f_1 are consecutive tabular values of the function, corresponding to arguments x_0, x_1, respectively; p is the given fraction of the argument interval

$$p=(x-x_0)/(x_1-x_0)$$

and f_p the required interpolate. In the present instance, we have

$f_0=.89717\ 4302 \qquad f_1=.89823\ 7113 \qquad p=.527$

The most convenient way to evaluate the formula on a desk calculating machine is to set f_0 and f_1 in turn on the keyboard, and carry out the multiplications by $1-p$ and p cumulatively; a partial check is then provided by the multiplier dial reading unity. We obtain

$$f_{.527}=(1-.527)(.89717\ 4302)+.527(.89823\ 7113)$$
$$=.89773\ 4403.$$

Since it is known that there is a possible error of 3×10^{-6} in the linear formula, we round off this result to .89773. The maximum possible error in this answer is composed of the error committed by the last rounding, that is, $.4403\times10^{-5}$, plus 3×10^{-6}, and so certainly cannot exceed $.8\times10^{-5}$.

(2) Lagrange's formula. In this example, the relevant formula is the 5-point one, given by

$$f=A_{-2}(p)f_{-2}+A_{-1}(p)f_{-1}+A_0(p)f_0+A_1(p)f_1+A_2(p)f_2$$

Tables of the coefficients $A_k(p)$ are given in chapter 25 for the range $p=0(.01)1$. We evaluate the formula for $p=.52$, .53 and .54 in turn. Again, in each evaluation we accumulate the $A_k(p)$ in the multiplier register since their sum is unity. We now have the following subtable.

x	$xe^xE_1(x)$		
7.952	.89772 9757		
		10622	
7.953	.89774 0379		−2
		10620	
7.954	.89775 0999		

The numbers in the third and fourth columns are the first and second differences of the values of $xe^xE_1(x)$ (see below); the smallness of the second difference provides a check on the three interpolations. The required value is now obtained by linear interpolation:

$$f_p=.3(.89772\ 9757)+.7(.89774\ 0379)$$
$$=.89773\ 7192.$$

In cases where the correct order of the Lagrange polynomial is not known, one of the preliminary interpolations may have to be performed with polynomials of two or more different orders as a check on their adequacy.

(3) Aitken's method of iterative linear interpolation. The scheme for carrying out this process in the present example is as follows:

n	x_n	$y_n=xe^xE_1(x)$	$y_{0,n}$	$y_{0,1,n}$	$y_{0,1,2,n}$	$y_{0,1,2,3,n}$	x_n-x
0	8.0	.89823 7113					.0473
1	7.9	.89717 4302	.89773 44034				−.0527
2	8.1	.89927 7888	.89774 48264	.89773 71499			.1473
3	7.8	.89608 8737	2 90220	2394	.89773 71938		−.1527
4	8.2	.90029 7306	4 98773	1216	16	89773 71930	.2473
5	7.7	.89497 9666	2 35221	2706	43	30	−.2527

Here

$$y_{0,n}=\frac{1}{x_n-x_0}\begin{vmatrix}y_0 & x_0-x\\ y_n & x_n-x\end{vmatrix}$$

$$y_{0,1,n}=\frac{1}{x_n-x_1}\begin{vmatrix}y_{0,1} & x_1-x\\ y_{0,n} & x_n-x\end{vmatrix}$$

$$y_{0,1,\ldots,m-1,m,n}=\frac{1}{x_n-x_m}\begin{vmatrix}y_{0,1,\ldots,m-1,m} & x_m-x\\ y_{0,1,\ldots,m-1,n} & x_n-x\end{vmatrix}$$

If the quantities x_n-x and x_m-x are used as multipliers when forming the cross-product on a desk machine, their accumulation $(x_n-x)-(x_m-x)$ in the multiplier register is the divisor to be used at that stage. An extra decimal place is usually carried in the intermediate interpolates to safeguard against accumulation of rounding errors.

The order in which the tabular values are used is immaterial to some extent, but to achieve the maximum rate of convergence and at the same time minimize accumulation of rounding errors, we begin, as in this example, with the tabular argument nearest to the given argument, then take the nearest of the remaining tabular arguments, and so on.

The number of tabular values required to achieve a given precision emerges naturally in the course of the iterations. Thus in the present example six values were used, even though it was known in advance that five would suffice. The extra row confirms the convergence and provides a valuable check.

(4) Difference formulas. We use the central difference notation (chapter 25),

x_0	f_0				
		$\delta f_{1/2}$			
x_1	f_1		$\delta^2 f_1$		
		$\delta f_{3/2}$		$\delta^3 f_{3/2}$	
x_2	f_2		$\delta^2 f_2$		$\delta^4 f_2$
		$\delta f_{5/2}$		$\delta^3 f_{5/2}$	
x_3	f_3		$\delta^2 f_3$		
		$\delta f_{7/2}$			
x_4	f_4				

Here

$$\delta f_{1/3}=f_1-f_0,\ \delta f_{3/2}=f_2-f_1,\ \ldots,$$
$$\delta^2 f_1=\delta f_{3/2}-\delta f_{1/2}=f_2-2f_1+f_0$$
$$\delta^3 f_{3/2}=\delta^2 f_2-\delta^2 f_1=f_3-3f_2+3f_1-f_0$$
$$\delta^4 f_2=\delta^3 f_{5/2}-\delta^3 f_{3/2}=f_4-4f_3+6f_2-4f_1+f_0$$

and so on.

In the present example the relevant part of the difference table is as follows, the differences being written in units of the last decimal place of the function, as is customary. The smallness of the high differences provides a check on the function values

x	$xe^xE_1(x)$	$\delta^2 f$	$\delta^4 f$
7.9	.89717 4302	−2 2754	−34
8.0	.89823 7113	−2 2036	−39

Applying, for example, Everett's interpolation formula

$$f_p=(1-p)f_0+E_2(p)\delta^2 f_0+E_4(p)\delta^4 f_0+\ldots+pf_1+F_2(p)\delta^2 f_1+F_4(p)\delta^4 f_1+\ldots$$

and taking the numerical values of the interpolation coefficients $E_2(p)$, $E_4(p)$, $F_2(p)$ and $F_4(p)$ from Table 25.1, we find that

$$10^9 f_{.527} = .473(89717\ 4302) + .061196(2\ 2754) - .012(34)$$
$$+ .527(89823\ 7113) + .063439(2\ 2036) - .012(39)$$
$$= 89773\ 7193.$$

We may notice in passing that Everett's formula shows that the error in a linear interpolate is approximately

$$E_2(p)\delta^2 f_0 + F_2(p)\delta^2 f_1 \approx \tfrac{1}{2}[E_2(p) + F_2(p)][\delta^2 f_0 + \delta^2 f_1]$$

Since the maximum value of $|E_2(p)+F_2(p)|$ in the range $0<p<1$ is $\frac{1}{8}$, the maximum error in a linear interpolate is approximately

$$\frac{1}{16}|\delta^2 f_0 + \delta^2 f_1|, \text{ that is, } \frac{1}{16}|f_2 - f_1 - f_0 + f_{-1}|.$$

(5) Taylor's series. In cases where the successive derivatives of the tabulated function can be computed fairly easily, Taylor's expansion

$$f(x) = f(x_0) + (x-x_0)\frac{f'(x_0)}{1!} + (x-x_0)^2\frac{f''(x_0)}{2!} + (x-x_0)^3\frac{f'''(x_0)}{3!} + \dots$$

can be used. We first compute as many of the derivatives $f^{(n)}(x_0)$ as are significant, and then evaluate the series for the given value of x. An advisable check on the computed values of the derivatives is to reproduce the adjacent tabular values by evaluating the series for $x=x_{-1}$ and x_1.

In the present example, we have

$$f(x) = xe^x E_1(x)$$
$$f'(x) = (1+x^{-1})f(x) - 1$$
$$f''(x) = (1+x^{-1})f'(x) - x^{-2}f(x)$$
$$f'''(x) = (1+x^{-1})f''(x) - 2x^{-2}f'(x) + 2x^{-3}f(x).$$

With $x_0=7.9$ and $x-x_0=.0527$ our computations are as follows; an extra decimal has been retained in the values of the terms in the series to safeguard against accumulation of rounding errors.

k	$f^{(k)}(x_0)/k!$	$(x-x_0)^k f^{(k)}(x_0)/k!$
0	.89717 4302	.89717 4302
1	.01074 0669	.00056 6033 3
2	−.00113 7621	−.00000 3159 5
3	.00012 1987	.00000 0017 9
		.89773 7194

5. Inverse Interpolation

With linear interpolation there is no difference in principle between direct and inverse interpolation. In cases where the linear formula provides an insufficiently accurate answer, two methods are available. We may interpolate directly, for example, by Lagrange's formula to prepare a new table at a fine interval in the neighborhood of the approximate value, and then apply accurate inverse linear interpolation to the subtabulated values. Alternatively, we may use Aitken's method or even possibly the Taylor's series method, with the roles of function and argument interchanged.

It is important to realize that the accuracy of an inverse interpolate may be very different from that of a direct interpolate. This is particularly true in regions where the function is slowly varying, for example, near a maximum or minimum. The maximum precision attainable in an inverse interpolate can be estimated with the aid of the formula

$$\Delta x \approx \Delta f / \frac{df}{dx}$$

in which Δf is the maximum possible error in the function values.

Example. Given $xe^x E_1(x) = .9$, find x from the table on page X.

(i) Inverse linear interpolation. The formula for p is

$$p = (f_p - f_0)/(f_1 - f_0).$$

In the present example, we have

$$p = \frac{.9 - .89927\ 7888}{.90029\ 7306 - .89927\ 7888} = \frac{72\ 2112}{101\ 9418} = .708357.$$

The desired x is therefore

$$x = x_0 + p(x_1 - x_0) = 8.1 + .708357(.1) = 8.17083\ 57$$

To estimate the possible error in this answer, we recall that the maximum error of direct linear interpolation in this table is $\Delta f = 3\times10^{-6}$. An approximate value for df/dx is the ratio of the first difference to the argument interval (chapter 25), in this case .010. Hence the maximum error in x is approximately $3\times10^{-6}/(.010)$, that is, .0003.

(ii) Subtabulation method. To improve the approximate value of x just obtained, we interpolate directly for $p=.70$, .71 and .72 with the aid of Lagrange's 5-point formula,

x	$xe^x E_1(x)$	δ	δ^2
8.170	.89999 3683		
		1 0151	
8.171	.90000 3834		−2
		1 0149	
8.172	.90001 3983		

Inverse linear interpolation in the new table gives

$$p = \frac{.9 - .89999\ 3683}{.00001\ 0151} = .6223$$

Hence $x = 8.17062\ 23$.

An estimate of the maximum error in this result is

$$\Delta f / \frac{df}{dx} \approx \frac{1\times10^{-9}}{.010} = 1\times10^{-7}$$

(iii) Aitken's method. This is carried out in the same manner as in direct interpolation.

n	$y_n = xe^x E_1(x)$	x_n	$x_{0,n}$	$x_{0,1,n}$	$x_{0,1,2,n}$	$x_{0,1,2,3,n}$	$y_n - y$
0	.90029 7306	8.2					.00029 7306
1	.89927 7888	8.1	8.17083 5712				−.00072 2112
2	.90129 6033	8.3	8.17023 1505	8.17061 9521			.00129 6033
3	.89823 7113	8.0	8.17113 8043	2 5948	8.17062 2244		−.00176 2887
4	.90227 4695	8.4	8.16992 9437	1 7335	415	8.17062 2318	.00227 4695
5	.89717 4302	7.9	8.17144 0382	2 8142	231	265	−.00282 5698

The estimate of the maximum error in this result is the same as in the subtabulation method. An indication of the error is also provided by the discrepancy in the highest interpolates, in this case $x_{0,1,2,3,4}$, and $x_{0,1,2,3,5}$.

6. Bivariate Interpolation

Bivariate interpolation is generally most simply performed as a sequence of univariate interpolations. We carry out the interpolation in one direction, by one of the methods already described, for several tabular values of the second argument in the neighborhood of its given value. The interpolates are differenced as a check, and interpolation is then carried out in the second direction.

An alternative procedure in the case of functions of a complex variable is to use the Taylor's series expansion, provided that successive derivatives of the function can be computed without much difficulty.

7. Generation of Functions from Recurrence Relations

Many of the special mathematical functions which depend on a parameter, called their index, order or degree, satisfy a linear difference equation (or recurrence relation) with respect to this parameter. Examples are furnished by the Legendre function $P_n(x)$, the Bessel function $J_n(x)$ and the exponential integral $E_n(x)$, for which we have the respective recurrence relations

$$(n+1)P_{n+1}-(2n+1)xP_n+nP_{n-1}=0$$

$$J_{n+1}-\frac{2n}{x}J_n+J_{n-1}=0$$

$$nE_{n+1}+xE_n=e^{-x}.$$

Particularly for automatic work, recurrence relations provide an important and powerful computing tool. If the values of $P_n(x)$ or $J_n(x)$ are known for two consecutive values of n, or $E_n(x)$ is known for one value of n, then the function may be computed for other values of n by successive applications of the relation. Since generation is carried out perforce with rounded values, it is vital to know how errors may be propagated in the recurrence process. If the errors do not grow relative to the size of the wanted function, the process is said to be stable. If, however, the relative errors grow and will eventually overwhelm the wanted function, the process is unstable.

It is important to realize that stability may depend on (i) the particular solution of the difference equation being computed; (ii) the values of x or other parameters in the difference equation; (iii) the direction in which the recurrence is being applied. Examples are as follows.

Stability—increasing n

$P_n(x)$, $P_n^m(x)$

$Q_n(x)$, $Q_n^m(x)$ $(x<1)$

$Y_n(x)$, $K_n(x)$

$J_{-n-\frac{1}{2}}(x)$, $I_{-n-\frac{1}{2}}(x)$

$E_n(x)$ $(n<x)$

Stability—decreasing n

$P_n(x)$, $P_n^m(x)$ $(x<1)$

$Q_n(x)$, $Q_n^m(x)$

$J_n(x)$, $I_n(x)$

$J_{n+\frac{1}{2}}(x)$, $I_{n+\frac{1}{2}}(x)$

$E_n(x)$ $(n>x)$

$F_n(\eta, \rho)$ (Coulomb wave function)

Illustrations of the generation of functions from their recurrence relations are given in the pertinent chapters. It is also shown that even in cases where the recurrence process is unstable, it may still be used when the starting values are known to sufficient accuracy.

Mention must also be made here of a refinement, due to J. C. P. Miller, which enables a recurrence process which is stable for decreasing n to be applied without any knowledge of starting values for large n. Miller's algorithm, which is well-suited to automatic work, is described in **19.28, Example 1.**

8. Acknowledgments

The production of this volume has been the result of the unrelenting efforts of many persons, all of whose contributions have been instrumental in accomplishing the task. The Editor expresses his thanks to each and every one.

The Ad Hoc Advisory Committee individually and together were instrumental in establishing the basic tenets that served as a guide in the formation of the entire work. In particular, special thanks are due to Professor Philip M. Morse for his continuous encouragement and support. Professors J. Todd and A. Erdélyi, panel members of the Conferences on Tables and members of the Advisory Committee have maintained an undiminished interest, offered many suggestions and carefully read all the chapters.

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M. Abramowitz.

Contents

1. Mathematical Constants

DAVID S. LIEPMAN[1]

Contents

[1] National Bureau of Standards.

TABLE 1. 1. MATHEMATICAL CONSTANTS

n(prime)	$\sqrt{n}$			
2	1. 4142	13562	37309	50488
3	1. 7320	50807	56887	72935
5	2. 2360	67977	49978	96964
7	2. 6457	51311	06459	05905
11	3. 3166	24790	35539	98491
13	3. 6055	51275	46398	92931
17	4. 1231	05625	61766	05498
19	4. 3588	98943	54067	35522
23	4. 7958	31523	31271	95416
29	5. 3851	64807	13450	40313
31	5. 5677	64362	83002	19221
37	6. 0827	62530	29821	96890
41	6. 4031	24237	43284	86865
43	6. 5574	38524	30200	06523
47	6. 8556	54600	40104	41249
53	7. 2801	09889	28051	82711
59	7. 6811	45747	86860	81758
61	7. 8102	49675	90665	43941
67	8. 1853	52771	87244	99700
71	8. 4261	49773	17635	86306
73	8. 5440	03745	31753	11679
79	8. 8881	94417	31558	88501
83	9. 1104	33579	14429	88819
89	9. 4339	81132	05660	38113
97	9. 8488	57801	79610	47217

$10^{1/2}$		3. 1622	77660	16837	93320	
$10^{1/3}$		2. 1544	34690	03188	37219	
$10^{1/4}$		1. 7782	79410	03892	28012	
$10^{1/5}$		1. 5848	93192	46111	34853	
$100^{1/3}$		4. 6415	88833	61277	88924	*
$100^{1/5}$		2. 5118	86431	50958	01112	
$1000^{1/4}$		5. 6234	13251	90349	08040	
$1000^{1/5}$		3. 9810	71705	53497	25077	*
$2^{1/3}$		1. 2599	21049	89487	31648	
$3^{1/3}$		1. 4422	49570	30740	83823	
$2^{1/4}$		1. 1892	07115	00272	10667	
$3^{1/4}$		1. 3160	74012	95249	24608	*
$2^{-1/2}$	(− 1)	7. 0710	67811	86547	52440	
$3^{-1/2}$	(− 1)	5. 7735	02691	89625	76451	
$5^{-1/2}$	(− 1)	4. 4721	35954	99957	93928	
$e^{\pi/2}$		4. 8104	77380	96535	16555	
$e^{\pi/4}$		2. 1932	80050	73801	54566	
$e^{-\pi/2}$	(− 1)	2. 0787	95763	50761	90855	
$e^{-\pi/4}$	(− 1)	4. 5593	81277	65996	23677	
$e^{1/2}$		1. 6487	21270	70012	81468	
$e^{-1/2}$	(− 1)	6. 0653	06597	12633	42360	
$e^{1/3}$		1. 3956	12425	08608	95286	
$e^{-1/3}$	(− 1)	7. 1653	13105	73789	25043	

n		e^n				
1		2. 7182	81828	45904	52353	60287
2		7. 3890	56098	93065	02272	30427
3	(1)	2. 0085	53692	31876	67740	92853
4	(1)	5. 4598	15003	31442	39078	11026
5	(2)	1. 4841	31591	02576	60342	11156
6	(2)	4. 0342	87934	92735	12260	83872
7	(3)	1. 0966	33158	42845	85992	63720
8	(3)	2. 9809	57987	04172	82747	43592
9	(3)	8. 1030	83927	57538	40077	09997
10	(4)	2. 2026	46579	48067	16516	95790

n		e^{-n}				
1	(− 1)	3. 6787	94411	71442	32159	55238
2	(− 1)	1. 3533	52832	36612	69189	39995
3	(− 2)	4. 9787	06836	78639	42979	34242
4	(− 2)	1. 8315	63888	87341	80293	71802
5	(− 3)	6. 7379	46999	08546	70966	36048
6	(− 3)	2. 4787	52176	66635	84230	45167
7	(− 4)	9. 1188	19655	54516	20800	31361
8	(− 4)	3. 3546	26279	02511	83882	13891
9	(− 4)	1. 2340	98040	86679	54949	76367
10	(− 5)	4. 5399	92976	24848	51535	59152

n		$e^{n\pi}$			
1	(1)	2. 3140	69263	27792	69006
2	(2)	5. 3549	16555	24764	73650
3	(4)	1. 2391	64780	79166	97482
4	(5)	2. 8675	13131	36653	29975
5	(6)	6. 6356	23999	34113	42333
6	(8)	1. 5355	29353	95446	69392
7	(9)	3. 5533	21280	84704	43597
8	(10)	8. 2226	31558	55949	95275
9	(12)	1. 9027	73895	29216	12917
10	(13)	4. 4031	50586	06320	29011
e^e	(1)	1. 5154	26224	14792	64190
e^γ		1. 7810	72417	99019	79852

n		$e^{-n\pi}$			
1	(− 2)	4. 3213	91826	37722	49774
2	(− 3)	1. 8674	42731	70798	88144
3	(− 5)	8. 0699	51757	03045	99239
4	(− 6)	3. 4873	42356	20899	54918
5	(− 7)	1. 5070	17275	39006	46107
6	(− 9)	6. 5124	12136	07990	07282
7	(−10)	2. 8142	68457	48555	27211
8	(−11)	1. 2161	55670	94093	08397
9	(−13)	5. 2554	85176	00644	85552
10	(−14)	2. 2711	01068	32409	38387
e^{-e}	(− 2)	6. 5988	03584	53125	37077
$e^{-\gamma}$	(− 1)	5. 6145	94835	66885	16982

n	$\ln n$				
2	0. 6931	47180	55994	53094	172321
3	1. 0986	12288	66810	96913	952452
4	1. 3862	94361	11989	06188	344642
5	1. 6094	37912	43410	03746	007593
6	1. 7917	59469	22805	50008	124774
7	1. 9459	10149	05531	33051	053527
8	2. 0794	41541	67983	59282	516964
9	2. 1972	24577	33621	93827	904905
10	2. 3025	85092	99404	56840	179915
11	2. 3978	95272	79837	05440	619436
13	2. 5649	49357	46153	67360	534874
17	2. 8332	13344	05621	60802	495346
19	2. 9444	38979	16644	04600	090274
23	3. 1354	94215	92914	96908	067528
29	3. 3672	95829	98647	40271	832720
31	3. 4339	87204	48514	62459	291643
37	3. 6109	17912	64422	44443	680957
41	3. 7135	72066	70430	78038	667634
43	3. 7612	00115	69356	24234	728425

n		$\log_{10} n$				
2	(−1)	3. 0102	99956	63981	19521	37389
3	(−1)	4. 7712	12547	19662	43729	50279
4	(−1)	6. 0205	99913	27962	39042	74778
5	(−1)	6. 9897	00043	36018	80478	62611
6	(−1)	7. 7815	12503	83643	63250	87668
7	(−1)	8. 4509	80400	14256	83071	22163
8	(−1)	9. 0308	99869	91943	58564	12167
9	(−1)	9. 5424	25094	39324	87459	00558
10		1. 0000	00000	00000	00000	00000
11		1. 0413	92685	15822	50407	50200
13		1. 1139	43352	30683	67692	06505
17		1. 2304	48921	37827	39285	40170
19		1. 2787	53600	95282	89615	36333
23		1. 3617	27836	01759	28788	67777
29		1. 4623	97997	89895	60873	32847
31		1. 4913	61693	83427	26796	66704
37		1. 5682	01724	06699	49968	08451
41		1. 6127	83856	71973	54945	09412
43		1. 6334	68455	57958	65264	05088

*See page II.

TABLE 1.1. MATHEMATICAL CONSTANTS—Continued

n		$\ln n$				
47		3.8501	47601	71005	85868	209507
53		3.9702	91913	55212	18341	444691
59		4.0775	37443	90571	94506	160504
61		4.1108	73864	17331	12487	513891
67		4.2046	92619	39096	60596	700720
71		4.2626	79877	04131	54213	294545
73		4.2904	59441	14839	11290	921089
79		4.3694	47852	46702	14941	729455
83		4.4188	40607	79659	79234	754722
89		4.4886	36369	73213	98383	178155
97		4.5747	10978	50338	28221	167216
$\ln\pi$		1.1447	29885	84940	01741	43427
$\ln\sqrt{2\pi}$	(−1)	9.1893	85332	04672	74178	03296

n		$\log_{10} n$				
47		1.6720	97857	93571	74644	14219
53		1.7242	75869	60078	90456	32992
59		1.7708	52011	64214	41902	60656
61		1.7853	29835	01076	70338	85749
67		1.8260	74802	70082	64341	49132
71		1.8512	58348	71907	52860	92829
73		1.8633	22860	12045	59010	74387
79		1.8976	27091	29044	14279	94821
83		1.9190	78092	37607	39038	32760
89		1.9493	90006	64491	27847	23543
97		1.9867	71734	26624	48517	84362
$\log_{10}\pi$	(−1)	4.9714	98726	94133	85435	12683
$\log_{10}e$	(−1)	4.3429	44819	03251	82765	11289

n		$n \ln 10$				
1		2.3025	85092	99404	56840	17991
2		4.6051	70185	98809	13680	35983
3		6.9077	55278	98213	70520	53974
4		9.2103	40371	97618	27360	71966
5	(1)	1.1512	92546	49702	28420	08996
6	(1)	1.3815	51055	79642	74104	10795
7	(1)	1.6118	09565	09583	19788	12594
8	(1)	1.8420	68074	39523	65472	14393
9	(1)	2.0723	26583	69464	11156	16192

n		$n\pi$				
1		3.1415	92653	58979	32384	62643
2		6.2831	85307	17958	64769	25287
3		9.4247	77960	76937	97153	87930
4	(1)	1.2566	37061	43591	72953	85057
5	(1)	1.5707	96326	79489	66192	31322
6	(1)	1.8849	55592	15387	59430	77586
7	(1)	2.1991	14857	51285	52669	23850
8	(1)	2.5132	74122	87183	45907	70115
9	(1)	2.8274	33388	23081	39146	16379

n		π^n				
1		3.1415	92653	58979	32384	62643
2		9.8696	04401	08935	86188	34491
3	(1)	3.1006	27668	02998	20175	47632
4	(1)	9.7409	09103	40024	37236	44033
5	(2)	3.0601	96847	85281	45326	27413
6	(2)	9.6138	91935	75304	43703	02194
7	(3)	3.0202	93227	77679	20675	14206
8	(3)	9.4885	31016	07057	40071	28576
9	(4)	2.9809	09933	34462	11666	50940
10	(4)	9.3648	04747	60830	20973	71669

n		π^{-n}				
1	(−1)	3.1830	98861	83790	67153	77675
2	(−1)	1.0132	11836	42337	77144	38795
3	(−2)	3.2251	53443	31994	89184	42205
4	(−2)	1.0265	98225	46843	35189	15278
5	(−3)	3.2677	63643	05338	54726	28250
6	(−3)	1.0401	61473	29585	22960	89838
7	(−4)	3.3109	36801	77566	76432	59528
8	(−4)	1.0539	03916	53493	66633	17287
9	(−5)	3.3546	80357	20886	91287	39854
10	(−5)	1.0678	27922	68615	33662	04078

$\pi/2$		1.5707	96326	79489	66192	31322
$\pi/3$		1.0471	97551	19659	77461	54214
$\pi/4$	(−1)	7.8539	81633	97448	30961	56608
$\pi^{1/2}$		1.7724	53850	90551	60272	98167
$\pi^{1/3}$		1.4645	91887	56152	32630	20143
$\pi^{1/4}$		1.3313	35363	80038	97127	97535
$\pi^{2/3}$		2.1450	29397	11102	56000	77444
$\pi^{3/4}$		2.3597	30492	41469	68875	78474
$\pi^{3/2}$		5.5683	27996	83170	78452	84818
π^{π}	(1)	2.2459	15771	83610	45473	42715
$(2\pi)^{1/2}$		2.5066	28274	63100	05024	15765
$(\pi/2)^{1/2}$		1.2533	14137	31550	02512	07883
$\pi(2)^{-1/2}$		2.2214	41469	07918	31235	07940

$3\pi/2$		4.7123	88980	38468	98576	93965
$4\pi/3$		4.1887	90204	78639	09846	16858
$\pi(2)^{1/2}$		4.4428	82938	15836	62470	15881
$\pi^{-1/2}$	(−1)	5.6418	95835	47756	28694	80795
$\pi^{-1/3}$	(−1)	6.8278	40632	55295	68146	70208
$\pi^{-1/4}$	(−1)	7.5112	55444	64942	48285	87030
$\pi^{-2/3}$	(−1)	4.6619	40770	35411	61438	19885
$\pi^{-3/4}$	(−1)	4.2377	72081	23757	59679	10077
$\pi^{-3/2}$	(−1)	1.7958	71221	25166	56168	90820
$\pi^{-\pi}$	(−2)	4.4525	26726	69229	06151	35273
$(2\pi)^{-1/2}$	(−1)	3.9894	22804	01432	67793	99461
$(2/\pi)^{1/2}$	(−1)	7.9788	45608	02865	35587	98921
$2^{1/2}/\pi$	(−1)	4.5015	81580	78553	03477	75996

$1r$	57.2957	79513	08232	08767	98155°
$1°$	0.0174	53292	51994	32957	69237r
$1'$	0.0002	90888	20866	57215	96154r
$1''$	0.0000	04848	13681	10953	59936r
γ	0.5772	15664	90153	28606	06512
$\ln\gamma$	−0.5495	39312	98164	48223	37662

$\Gamma(1/2)$	1.7724	53850	905516
$\Gamma(1/3)$	2.6789	38534	707748
$\Gamma(2/3)$	1.3541	17939	426400
$\Gamma(1/4)$	3.6256	09908	221908
$\Gamma(3/4)$	1.2254	16702	465178
$\Gamma(4/3)$	0.8929	79511	569249
$\Gamma(5/3)$	0.9027	45292	950934
$\Gamma(5/4)$	0.9064	02477	055477
$\Gamma(7/4)$	0.9190	62526	848883
$\ln\Gamma(1/3)$	0.9854	20646	927767
$\ln\Gamma(2/3)$	0.3031	50275	147523
$\ln\Gamma(1/4)$	1.2880	22524	698077
$\ln\Gamma(3/4)$	0.2032	80951	431296

$1/\Gamma(1/2)$	0.5641	89583	547756
$1/\Gamma(1/3)$	0.3732	82173	907395
$1/\Gamma(2/3)$	0.7384	88111	621648
$1/\Gamma(1/4)$	0.2758	15662	830209
$1/\Gamma(3/4)$	0.8160	48939	098263
$1/\Gamma(4/3)$	1.1198	46521	722186
$1/\Gamma(5/3)$	1.1077	32167	432472
$1/\Gamma(5/4)$	1.1032	62651	320837
$1/\Gamma(7/4)$	1.0880	65252	131017
$\ln\Gamma(4/3)$	−0.1131	91641	740343
$\ln\Gamma(5/3)$	−0.1023	14832	960640
$\ln\Gamma(5/4)$	−0.0982	71836	421813
$\ln\Gamma(7/4)$	−0.0844	01121	020486

*See page II.

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