$2.95

HYDRODYNAMICS

Sir Horace Lamb

". . . has long been the chief storehouse of information of all workers in hydrodynamics . . ." Nature

dixon

PREFACE

THIS may be regarded as the sixth edition of a *Treatise on the Mathematical Theory of the Motion of Fluids*, published in 1879. Subsequent editions, largely remodelled and extended, have appeared under the present title.

In this issue no change has been made in the general plan and arrangement, but the work has again been revised throughout, some important omissions have been made good, and much new matter has been introduced.

The subject has in recent years received considerable developments, in the theory of the tides for instance, and in various directions bearing on the problems of aeronautics, and it is interesting to note that the "classical" Hydrodynamics, often referred to with a shade of depreciation, is here found to have a widening field of practical applications. Owing to the elaborate nature of some of these researches it has not always been possible to fit an adequate account of them into the frame of this book, but attempts have occasionally been made to give some indication of the more important results, and of the methods employed.

As in previous editions, pains have been taken to make due acknowledgment of authorities in the footnotes, but it appears necessary to add that the original proofs have often been considerably modified in the text.

I have again to thank the staff of the University Press for much valued assistance during the printing.

HORACE LAMB

April 1932

CONTENTS

CHAPTER I

THE EQUATIONS OF MOTION

CHAPTER II

INTEGRATION OF THE EQUATIONS IN SPECIAL CASES

CHAPTER III

IRROTATIONAL MOTION

CHAPTER IV

MOTION OF A LIQUID IN TWO DIMENSIONS

CHAPTER V

IRROTATIONAL MOTION OF A LIQUID: PROBLEMS IN THREE DIMENSIONS

CHAPTER VI

ON THE MOTION OF SOLIDS THROUGH A LIQUID: DYNAMICAL THEORY

CHAPTER VII

VORTEX MOTION

CHAPTER VIII

TIDAL WAVES

CHAPTER IX

SURFACE WAVES

CHAPTER X

WAVES OF EXPANSION

CHAPTER XI

VISCOSITY

CHAPTER XII

ROTATING MASSES OF LIQUID

HYDRODYNAMICS

CHAPTER I

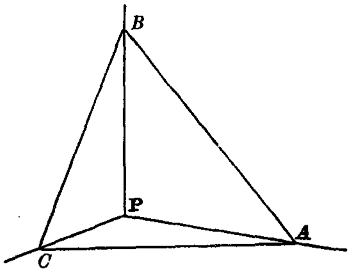
THE EQUATIONS OF MOTION

1. THE following investigations proceed on the assumption that the matter with which we deal may be treated as practically continuous and homogeneous in structure; *i.e.* we assume that the properties of the smallest portions into which we can conceive it to be divided are the same as those of the substance in bulk.

The fundamental property of a fluid is that it cannot be in equilibrium in a state of stress such that the mutual action between two adjacent parts is oblique to the common surface. This property is the basis of Hydrostatics, and is verified by the complete agreement of the deductions of that science with experiment. Very slight observation is enough, however, to convince us that oblique stresses may exist in fluids *in motion*. Let us suppose for instance that a vessel in the form of a circular cylinder, containing water (or other liquid), is made to rotate about its axis, which is vertical. If the angular velocity of the vessel be constant, the fluid is soon found to be rotating with the vessel as one solid body. If the vessel be now brought to rest, the motion of the fluid continues for some time, but gradually subsides, and at length ceases altogether; and it is found that during this process the portions of fluid which are further from the axis lag behind those which are nearer, and have their motion more rapidly checked. These phenomena point to the existence of mutual actions between contiguous elements which are partly tangential to the common surface. For if the mutual action were everywhere wholly normal, it is obvious that the moment of momentum, about the axis of the vessel, of any portion of fluid bounded by a surface of revolution about this axis, would be constant. We infer, moreover, that these tangential stresses are not called into play so long as the fluid moves as a solid body, but only whilst a change of shape of some portion of the mass is going on, and that their tendency is to oppose this change of shape.

2. It is usual, however, in the first instance to neglect the tangential stresses altogether. Their effect is in many practical cases small, and, independently of this, it is convenient to divide the not inconsiderable difficulties of our subject by investigating first the effects of purely normal stress. The further consideration of the laws of tangential stress is accordingly deferred till Chapter XI.

If the stress exerted across any small plane area situate at a point P of the fluid be wholly normal, its intensity (per unit area) is the same for all aspects of the plane. The following proof of this theorem is given here for purposes of reference. Through P draw three straight lines PA, PB, PC mutually at right angles, and let a plane whose direction-cosines relatively to these lines are l, m, n, passing infinitely close to P, meet them in A, B, C. Let p, p_1, p_2, p_3 denote the intensities of the stresses* across the faces ABC, PBC, PCA, PAB, respectively, of the tetrahedron $PABC$. If Δ be the area of the first-mentioned face, the areas of the others are, in order, $l\Delta$, $m\Delta$, $n\Delta$. Hence if we form the equation of motion of the tetrahedron parallel to PA we have $p_1 \, . \, l\Delta = pl \, . \, \Delta$, where we have omitted the terms which express the rate of change of momentum, and the component of the extraneous forces, because they are ultimately proportional to the mass of the tetrahedron, and therefore of the third order of small linear quantities, whilst the terms retained are of the second. We have then, ultimately, $p = p_1$, and similarly $p = p_2 = p_3$, which proves the theorem.



3. The equations of motion of a fluid have been obtained in two different forms, corresponding to the two ways in which the problem of determining the motion of a fluid mass, acted on by given forces and subject to given conditions, may be viewed. We may either regard as the object of our investigations a knowledge of the velocity, the pressure, and the density, at all points of space occupied by the fluid, for all instants; or we may seek to determine the history of every particle. The equations obtained on these two plans are conveniently designated, as by German mathematicians, the 'Eulerian' and the 'Lagrangian' forms of the hydrokinetic equations, although both forms are in reality due to Euler†.

The Eulerian Equations.

4. Let u, v, w be the components, parallel to the co-ordinate axes, of the velocity at the point (x, y, z) at the time t. These quantities are then functions of the independent variables x, y, z, t. For any particular value of t they define the motion at that instant at all points of space occupied by

* Reckoned positive when pressures, negative when tensions. Most fluids are, however, incapable under ordinary conditions of supporting more than an exceedingly slight degree of tension, so that p is nearly always positive.

† "Principes généraux du mouvement des fluides," *Hist. de l'Acad. de Berlin*, 1755.
"De principiis motus fluidorum," *Novi Comm. Acad. Petrop.* xiv. 1 (1759).
Lagrange gave three investigations of the equations of motion; first, incidentally, in

the fluid; whilst for particular values of x, y, z they give the history of what goes on at a particular place.

We shall suppose, for the most part, not only that u, v, w are finite and continuous functions of x, y, z, but that their space-derivatives of the first order ($\partial u/\partial x$, $\partial v/\partial x$, $\partial w/\partial x$, &c.) are everywhere finite*; we shall understand by the term 'continuous motion,' a motion subject to these restrictions. Cases of exception, if they present themselves, will require separate examination. In continuous motion, as thus defined, the relative velocity of any two neighbouring particles P, P' will always be infinitely small, so that the line PP' will always remain of the same order of magnitude. It follows that if we imagine a small closed surface to be drawn, surrounding P, and suppose it to move with the fluid, it will always enclose the same matter. And *any* surface whatever, which moves with the fluid, completely and permanently separates the matter on the two sides of it.

5. The values of u, v, w for successive values of t give as it were a series of pictures of consecutive stages of the motion, in which however there is no immediate means of tracing the identity of any one particle.

To calculate the rate at which any function $F(x, y, z, t)$ varies for a moving particle, we may remark that at the time $t + \delta t$ the particle which was originally in the position (x, y, z) is in the position $(x + u\,\delta t,\ y + v\,\delta t,\ z + w\,\delta t)$, so that the corresponding value of F is

$$F(x + u\,\delta t,\ y + v\,\delta t,\ z + w\,\delta t,\ t + \delta t) = F + u\,\delta t\frac{\partial F}{\partial x} + v\,\delta t\frac{\partial F}{\partial y} + w\,\delta t\frac{\partial F}{\partial z} + \delta t\,\frac{\partial F}{\partial t}.$$

If, after Stokes, we introduce the symbol D/Dt to denote a differentiation following the motion of the fluid, the new value of F is also expressed by $F + DF/Dt\,.\,\delta t$, whence

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z}. \quad\text{.....................(1)}$$

6. To form the dynamical equations, let p be the pressure, ρ the density, X, Y, Z the components of the extraneous forces per unit mass, at the point (x, y, z) at the time t. Let us take an element having its centre at (x, y, z), and its edges δx, δy, δz parallel to the rectangular co-ordinate axes. The rate at which the x-component of the momentum of this element is increasing is $\rho\,\delta x\,\delta y\,\delta z\,Du/Dt$; and this must be equal to the x-component of the forces

connection with the principle of Least Action, in the *Miscellanea Taurinensia*, ii. (1760) [*Oeuvres*, Paris, 1867–92, i.]; secondly in his "Mémoire sur la Théorie du Mouvement des Fluides," *Nouv. mém. de l'Acad. de Berlin*, 1781 [*Oeuvres*, iv.]; and thirdly in the *Mécanique Analytique*. In this last exposition he starts with the second form of the equations (Art. 14, below), but translates them at once into the 'Eulerian' notation.

* It is important to bear in mind, with a view to some later developments under the head of Vortex Motion, that these derivatives need not be assumed to be continuous.

acting on the element. Of these the extraneous forces give $\rho\,\delta x\,\delta y\,\delta z\,X$. The pressure on the yz-face which is nearest the origin will be ultimately

$$(p - \tfrac{1}{2}\partial p/\partial x\,.\,\delta x)\,\delta y\,\delta z\,*,$$

that on the opposite face

$$(p + \tfrac{1}{2}\partial p/\partial x\,.\,\delta x)\,\delta y\,\delta z.$$

The difference of these gives a resultant $-\partial p/\partial x\,.\,\delta x\,\delta y\,\delta z$ in the direction of x-positive. The pressures on the remaining faces are perpendicular to x. We have then

$$\rho\,\delta x\,\delta y\,\delta z\,\frac{Du}{Dt} = \rho\,\delta x\,\delta y\,\delta z\,X - \frac{\partial p}{\partial x}\,\delta x\,\delta y\,\delta z.$$

Substituting the value of Du/Dt from (1), and writing down the symmetrical equations, we have

$$\left.\begin{aligned}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= X - \frac{1}{\rho}\frac{\partial p}{\partial x},\\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= Y - \frac{1}{\rho}\frac{\partial p}{\partial y},\\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= Z - \frac{1}{\rho}\frac{\partial p}{\partial z}\end{aligned}\right\} \quad \text{..............(2)}$$

7. To these dynamical equations we must join, in the first place, a certain kinematical relation between u, v, w, ρ, obtained as follows.

If Q be the volume of a moving element, we have, on account of the constancy of mass,

$$\frac{D\,.\,\rho Q}{Dt} = 0,$$

or

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \frac{1}{Q}\frac{DQ}{Dt} = 0. \quad \text{..............................(1)}$$

To calculate the value of $1/Q\,.\,DQ/Dt$, let the element in question be that which at time t fills the rectangular space $\delta x\,\delta y\,\delta z$ having one corner P at (x, y, z), and the edges PL, PM, PN (say) parallel to the co-ordinate axes. At time $t + \delta t$ the same element will form an oblique parallelepiped, and since the velocities of the particle L relative to the particle P are $\partial u/\partial x\,.\,\delta x$, $\partial v/\partial x\,.\,\delta x$, $\partial w/\partial x\,.\,\delta x$, the projections of the edge PL on the co-ordinate axes become, after the time δt,

$$\left(1 + \frac{\partial u}{\partial x}\,\delta t\right)\delta x, \quad \frac{\partial v}{\partial x}\,\delta t\,.\,\delta x, \quad \frac{\partial w}{\partial x}\,\delta t\,.\,\delta x,$$

respectively. To the first order in δt, the length of this edge is now

$$\left(1 + \frac{\partial u}{\partial x}\,\delta t\right)\delta x,$$

and similarly for the remaining edges. Since the angles of the parallelepiped

* It is easily seen, by Taylor's theorem, that the mean pressure over any face of the element $\delta x\,\delta y\,\delta z$ may be taken to be equal to the pressure at the centre of that face.

differ infinitely little from right angles, the volume is still given, to the first order in δt, by the product of the three edges, *i.e.* we have

$$Q+\frac{DQ}{Dt}\delta t=\left\{1+\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\delta t\right\}\delta x\,\delta y\,\delta z,$$

or

$$\frac{1}{Q}\frac{DQ}{Dt}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}. \qquad \text{.........................(2)}$$

Hence (1) becomes

$$\frac{D\rho}{Dt}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0. \qquad \text{.....................(3)}$$

This is called the 'equation of continuity.'

The expression

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}, \qquad \text{..............................(4)}$$

which, as we have seen, measures the rate of dilatation of the fluid at the point (x, y, z), is conveniently called the 'expansion' at that point. From a more general point of view the expression (4) is called the 'divergence' of the vector (u, v, w); it is often denoted briefly by

$$\text{div}\,(u, v, w).$$

The preceding investigation is substantially that given by Euler*. Another, and now more usual, method of obtaining the equation of continuity is, instead of following the motion of a fluid element, to fix the attention on an element $\delta x \delta y \delta z$ of space, and to calculate the change produced in the included mass by the flux across the boundary. If the centre of the element be at (x, y, z), the amount of matter which per unit time enters it across the yz-face nearest the origin is

$$\left(\rho u-\tfrac{1}{2}\frac{\partial\,.\,\rho u}{\partial x}\delta x\right)\delta y\,\delta z,$$

and the amount which leaves it by the opposite face is

$$\left(\rho u+\tfrac{1}{2}\frac{\partial\,.\,\rho u}{\partial x}\delta x\right)\delta y\,\delta z.$$

The two faces together give a gain

$$-\frac{\partial\,.\,\rho u}{\partial x}\delta x\,\delta y\,\delta z,$$

per unit time. Calculating in the same way the effect of the flux across the remaining faces, we have for the total gain of mass, per unit time, in the space $\delta x\,\delta y\,\delta z$, the formula

$$-\left(\frac{\partial\,.\,\rho u}{\partial x}+\frac{\partial\,.\,\rho v}{\partial y}+\frac{\partial\,.\,\rho w}{\partial z}\right)\delta x\,\delta y\,\delta z.$$

Since the quantity of matter in any region can vary only in consequence of the flux across the boundary, this must be equal to

$$\frac{\partial}{\partial t}(\rho\,\delta x\,\delta y\,\delta z),$$

* *l.c. ante* p. 2.

whence we get the equation of continuity in the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial \, . \, \rho u}{\partial x} + \frac{\partial \, . \, \rho v}{\partial y} + \frac{\partial \, . \, \rho w}{\partial z} = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

8. It remains to put in evidence the physical properties of the fluid, so far as these affect the quantities which occur in our equations.

In an 'incompressible' fluid, or liquid, we have $D\rho/Dt = 0$, in which case the equation of continuity takes the simple form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

It is not assumed here that the fluid is of *uniform* density, though this is of course by far the most important case.

If we wish to take account of the slight compressibility of actual liquids, we shall have a relation of the form

$$p = \kappa(\rho - \rho_0)/\rho_0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

or

$$\rho/\rho_0 = 1 + p/\kappa, \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

where κ denotes what is called the 'elasticity of volume.'

In the case of a gas whose temperature is uniform and constant we have the 'isothermal' relation

$$p/p_0 = \rho/\rho_0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

where p_0, ρ_0 are any pair of corresponding values for the temperature in question.

In most cases of motion of gases, however, the temperature is not constant, but rises and falls, for each element, as the gas is compressed or rarefied. When the changes are so rapid that we can ignore the gain or loss of heat by an element due to conduction and radiation, we have the 'adiabatic' relation

$$p/p_0 = (\rho/\rho_0)^\gamma, \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

where p_0 and ρ_0 are any pair of corresponding values for the element considered. The constant γ is the ratio of the two specific heats of the gas; for atmospheric air, and some other gases, its value is about 1·408.

9. At the boundaries (if any) of the fluid, the equation of continuity is replaced by a special surface-condition. Thus at a *fixed* boundary, the velocity of the fluid perpendicular to the surface must be zero, *i.e.* if l, m, n be the direction-cosines of the normal,

$$lu + mv + nw = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Again at a surface of discontinuity, *i.e.* a surface at which the values of u, v, w change abruptly as we pass from one side to the other, we must have

$$l(u_1 - u_2) + m(v_1 - v_2) + n(w_1 - w_2) = 0, \quad \ldots\ldots\ldots\ldots(2)$$

where the suffixes are used to distinguish the values on the two sides. The same relation must hold at the common surface of a fluid and a moving solid.