PATH INTEGRALS

in

Quantum Mechanics
Statistics
and
Polymer Physics

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Published by

World Scientific Publishing Co. Pte. Ltd.

P O Box 128, Farrer Road, Singapore 9128

USA office: 687 Hartwell Street, Teaneck, NJ 07666

UK office: 73 Lynton Mead, Totteridge, London N20 8DH

Library of Congress Cataloging-in-Publication Data

Kleinert, Hagen.
Path integrals in quantum mechanics, statistics, and polymer physics / Hagen Kleinert.
p. cm.
Includes bibliographical references and index.
ISBN 9810201966. -- ISBN 9810201974 (pbk.)
1. Path integrals. 2. Quantum theory. 3. Statistical physics. 4. Polymers. I. Title.
QC174.17.P27K54 1990
530.1'2--dc20 90-45950
CIP

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Ay, call it holy ground,
The soil where first they trod!
F.D. HEMANS (1793-1835),
Landing of the Pilgrim Fathers

Chapter 1

Fundamentals

Before turning to the actual subject of this text, path integrals, it may be useful to recall some of the theoretical background underlying the theory to be developed.[1]

1.1 Classical Mechanics

A classical mechanical system is described by a set of generalized coordinates $q_1, \ldots, q_N$, the associated velocities, $\dot{q}_1, \ldots, \dot{q}_N$, and a Langrangian

$$L(q_i(t), \dot{q}_i(t), t). \tag{1.1}$$

The dot denotes the time derivative d/dt. The Lagrangian governs the dynamics and is, at most, a quadratic function of $\dot{q}_i$. The time integral

$$\mathcal{A}[q_i] = \int_{t_a}^{t_b} dt\, L(q_i(t), \dot{q}_i(t), t) \tag{1.2}$$

over the Lagrangian along an arbitrary path $q_i(t)$ is called the *action* of this path. The path $q_i(t)$ that is actually chosen by the system as a function of time is called the *classical path*, $q_i^{cl}(t)$. It has the property of extremizing the action in comparison with all neighboring paths

$$q_i(t) = q_i^{cl}(t) + \delta q_i(t) \tag{1.3}$$

with fixed end points $q(t_b)$, $q(t_a)$. To express this property formally one introduces the *variation* of the action as the linear term in $\delta q_i(t)$ of the change of $\mathcal{A}[q_i]$:

$$\delta \mathcal{A}[q_i] \equiv \{\mathcal{A}[q_i + \delta q_i] - \mathcal{A}[q_i]\}_{lin} \tag{1.4}$$

[1]The reader familiar with classical and quantum mechanics may skip the first four sections and start with Section 1.5.

The extremal principle for the classical path is then

$$\delta\mathcal{A}[q_i]\Big|_{q_i(t)=q_i^{cl}(t)} = 0 \tag{1.5}$$

for all variations of the path around the classical path, $\delta q_i(t) \equiv q_i(t) - q_i^{cl}(t)$, which vanish at the end points:

$$\delta q_i(t_a) = \delta q_i(t_b) = 0. \tag{1.6}$$

Since the action is a time integral of a Lagrangian, this extremality property can be phrased in terms of differential equations. Let us calculate the variation of $\mathcal{A}[q_i]$ explicitly,

$$\begin{aligned}
\delta\mathcal{A}[q_i] &= \{\mathcal{A}[q_i + \delta q_i] - \mathcal{A}[q_i]\}_{lin} \\
&= \int_{t_a}^{t_b} dt\ \{L(q_i(t) + \delta q_i(t), \dot{q}_i(t) + \delta\dot{q}_i(t), t) - L(q_i(t), \dot{q}_i(t), t)\}_{lin} \\
&= \int_{t_a}^{t_b} dt \left\{\frac{\partial L}{\partial q_i}\delta q_i(t) + \frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i(t)\right\} \\
&= \int_{t_a}^{t_b} dt \left\{\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right\}\delta q_i(t) + \frac{\partial L}{\partial \dot{q}_i}\delta q_i(t)\Big|_{t_a}^{t_b}.
\end{aligned} \tag{1.7}$$

The last expression arises by partially integrating the $\delta\dot{q}_i$ term. Here, as in the entire text. repeated indices are understood to be summed (*Einstein's summation convention*). The end point terms (*surface or boundary terms*) at t_a and t_b may be dropped, due to (1.6). Thus we find, for the classical orbit $q_i^{cl}(t)$, the so-called *Euler-Lagrange equations*:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}. \tag{1.8}$$

There is an alternative formulation of classical dynamics which is based on a Legendre transformed function of the Lagrangian called the *Hamiltonian*

$$H(p_i(t), q_i(t), t) \equiv \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i(t) - L(q_i(t), \dot{q}_i(t), t). \tag{1.9}$$

Its value at any time is identified with the energy of the system. According to the general theory of Legendre transformations,[2] the natural variables in H are no longer $q_i(t)$ and $\dot{q}_i(t)$, but $q_i(t)$ and the generalized momenta $p_i(t)$ defined by the equations

$$p_i(t) \equiv \frac{\partial}{\partial \dot{q}_i}L(q_i(t), \dot{q}_i(t), t). \tag{1.10}$$

[2] **For an elementary introduction see the book H.B. Callan, *Classical Thermodynamics*, John Wiley and Sons, New York, 1960.**

In order to specify the Hamiltonian $H(p_i(t), q_i(t), t)$ in terms of its proper variables $p_i(t)$, $q_i(t)$, the equations (1.10) for $p_i(t)$ have to be solved for $\dot{q}_i(t)$,

$$\dot{q}_i(t) = v_i(p_i(t), q_i(t), t), \tag{1.11}$$

and inserted into (1.9), giving H as

$$H(p_i(t), q_i(t), t) = p_i(t)v_i(p_i(t), q_i(t), t) - L(q_i(t), v_i(p_i(t), q_i(t)), t). \tag{1.12}$$

Expressed in terms of H, the action becomes a functional of $p_i(t)$ and $q_i(t)$,

$$\mathcal{A}[p, q] = \int_{t_a}^{t_b} dt \left[p_i(t)\dot{q}_i(t) - H(p_i(t), q_i(t), t)\right]. \tag{1.13}$$

This is the so-called *canonical form* of the action. The classical orbits, now specified by $p_i^{cl}(t)$, $q_i^{cl}(t)$, extremize the action in comparison with all neighboring orbits when $q_i(t)$ are varied at fixed end points [see (1.3), (1.6)], and $p_i(t)$ without restriction:

$$\begin{aligned} q_i(t) &= q_i^{cl}(t) + \delta q_i(t), \qquad \delta q_i(t_a) = \delta q_i(t_b) = 0, \\ p_i(t) &= p_i^{cl}(t) + \delta p_i(t). \end{aligned} \tag{1.14}$$

This gives a variation

$$\begin{aligned} \delta\mathcal{A}[p, q] &= \int_{t_a}^{t_b} dt \left[\delta p_i(t)\dot{q}_i(t) + p_i(t)\delta\dot{q}_i(t) - \frac{\partial H}{\partial p_i}\delta p_i - \frac{\partial H}{\partial q_i}\delta q_i\right] \\ &= \int_{t_a}^{t_b} dt \left\{\left(\dot{q}_i(t) - \frac{\partial H}{\partial p_i}\right)\delta p_i - \left(\dot{p}_i(t) + \frac{\partial H}{\partial q_i}\right)\delta q_i\right\} \\ &\qquad + p_i(t)\delta q_i(t)\Big|_{t_b}^{t_b} \end{aligned} \tag{1.15}$$

From this we find the *Hamilton equations* of motion for the classical orbits,

$$\begin{aligned} \dot{p}_i &= -\frac{\partial H}{\partial q_i}, \\ \dot{q}_i &= \frac{\partial H}{\partial p_i}. \end{aligned} \tag{1.16}$$

These agree with the Euler-Lagrange equations (1.8) via (1.10), as can easily be verified.

The $2N$-dimensional space of all p_i and q_i is called the *phase space*. An arbitrary function

$$F(p_i(t), q_i(t), t) \tag{1.17}$$