Finite Element Analysis in Fluid Dynamics

T. J. CHUNG

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PREFACE

This book is an outgrowth of a series of lectures presented to the graduate students at The University of Alabama in Huntsville over a period of years. It begins with mathematical preliminaries and develops into a finite element analysis for solving partial differential equations of boundary and initial value problems in fluid dynamics. The book is intended for not only the uninitiated student but also for the scientist and engineer in practice.

The initial and boundary value problems in continuous media have been the subject of intensive study for centuries. Although many analytical and numerical methods of solution have been developed extensively, there still remains unsolved a considerable number of important problems. The electronic computer in modern times led to the new ideas in the approximation theory. The finite element method with the use of a computer has rapidly become one of the most powerful tools in solving the complex problems of continuous media in general.

The primary objective here is to learn how to *solve* practical problems in fluid dynamics. The beginner, upon thorough comprehension of the solution procedures in some selected subjects, will then find the rest of the topics self-explanatory, in which details are no longer needed and thus only the essence is presented. The mathematical properties of finite elements undoubtedly constitute an important part of our study. Thus the secondary objective is to provide a mathematical treatment with which the analyst can establish validity of his calculations.

A review of the historical background of the finite element method is given in Chapter 1. An elementary account of functional analysis, which is the foundation for the finite element error estimates, is also presented. However, a study of this subject is not a prerequisite for understanding the rest of this book if one just wishes to learn how to solve a problem. Following discussions of variational principles and weighted residual methods, we demonstrate simple one-dimensional finite element solutions for the benefit of the uninitiated student. Chapter 2 discusses various types of finite elements grouped into one-, two-, and three-dimensional and axisymmetric geometries, along with local and global interpolation functions and dual spaces. Assembly of local equations into a global form, imposition of boundary conditions, and solution of nonlinear equations and time-dependent problems are presented in Chapter 3. Error estimates for linear problems are also included.

The topics of fluid dynamics begin with Chapter 4 where basic fluid dynamics equations are reviewed. Incompressible and compressible flows are covered in Chapters 5 and 6, respectively. Here solutions of the Laplace equation for two-dimensional domain and Stokesian equation for axisymmetric geometry are highlighted by complete details using triangular and isoparametric elements. Then various alternative formulations of steady and unsteady incompressible flow problems follow, including the velocity, pressure, stream function, and vorticity as variables. Error analyses from the concept of Sobolev spaces are discussed where appropriate. Problems of free surface, eigenvalues and eigenfunctions for wave motion, turbulent boundary layers, three-dimensional flow, boundary singularities, and some discussions of finite elements versus finite differences are also included. In the case of a compressible flow with temperature and density as additional variables, the methodology as used for incompressible flow can be applied, and thus unnecessary repetitions are avoided. Both viscous and inviscid compressible flows are discussed. Some recent developments of transonic aerodynamics are summarized. Finally, selected miscellaneous topics such as diffusion, magnetohydrodynamics, and rarefied gas dynamics are presented in Chapter 7.

The finite element method is rapidly expanding in its scope of applications. In the meantime, commitments of the mathematicians, much to the surprise of the engineers, have provided new meaning, momentum, and confidence to this new field of research—the finite element analysis. It is for the unification of our knowledge sought by both engineers and mathematicians and for the spirit of teamwork that the present book is intended.

In the writing of this book, I am indebted to a countless number of authors of pioneering works from which I have freely drawn some of the materials and viewpoints. Professor J. Tinsley Oden reviewed the manuscript and offered invaluable suggestions for improvement. I wish to express my deepest appreciation to him. Thanks are also due to my former and present students who assisted in the solution of example problems. Among them are Dr. J. K. Lee, Messrs. C. G. Hooks, J. N. Chiou, and R. H. Rush. I owe a particular debt of gratitude to Mrs. Barbara Sweeney, who provided excellent service in computer programming. My thanks are further extended to Professor S. T. Wu for reviewing a portion of the manuscript, and to Professors J. J. Brainerd and C. C. Shih, among other colleagues, who shared useful discussions with me.

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University of Alabama, Huntsville
March 1977

T. J. Chung

CONTENTS

CHAPTER
ONE

INTRODUCTION

1-1 GENERAL

The finite element method is an approximate method of solving differential equations of boundary and/or initial value problems in engineering and mathematical physics. In this method, a continuum is divided into many small elements of convenient shapes—triangular, quadrilateral, etc. Choosing suitable points called "nodes" within the elements, the variable in the differential equation is written as a linear combination of appropriately selected interpolation functions and the values of the variable or its various derivatives specified at the nodes. Using variational principles or weighted residual methods, the governing differential equations are transformed into "finite element equations" governing all isolated elements. These local elements are finally collected together to form a global system of differential or algebraic equations with proper boundary and/or initial conditions imposed. The nodal values of the variable are determined from this system of equations.

The finite element method was originally developed by aircraft structural engineers in the 1950's to analyze large systems of structural elements in the aircraft. Turner, Clough, Martin, and Topp [1956] presented the first paper on the subject, followed by Clough [1960] and Argyris [1963], among others. Application of the finite element method to nonstructural problems such as fluid flows and electromagnetism was initiated by Zienkiewicz [1965], and applications

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to a wide class of problems of interest in nonlinear mechanics was contributed by Oden [1972].

The close relationship of finite element analysis to the classical variational concept of the Rayleigh–Ritz method [Rayleigh, 1877; Ritz, 1909] or the weighted residual methods modeled after the well-known method of Galerkin [1915] has established the finite element method as an important branch of approximation theory. In recent years various authors have contributed to the development of the mathematical theory of finite elements. Among them are Babuska and Aziz [1972], Ciarlet and Raviart [1972], Aubin [1972], Strang and Fix [1972], and Oden and Reddy [1976]. These recent developments have been greatly influenced by the pioneering works of Lions and Magenes [1972, English translation].

Today various theories of fluid behavior are available which encompass virtually any type of phenomena of much immediate practical interest. However, there remains a surprising number of unsolved important problems in fluid dynamics due to difficulties encountered in most of the conventional analytical and numerical methods. In fluid dynamics, a choice of Eulerian coordinates renders the resulting governing equations nonlinear in general, and thus analytically difficult to solve. The most widely used numerical method of overcoming these difficulties has been the method of finite differences [Richtmyer and Morton, 1967; Roache, 1972] in which the partial derivatives in the governing equations are replaced by finite difference quotients. Another numerical method in limited use is the particle-in-cell method [Evans and Harlow, 1957], in which a system of cells is constructed so as to define the position of fluid particles in terms of these cells, and each cell is characterized by a set of variables describing the mean components of velocity, internal energy, density, and pressure in the cell. Among other popular methods are the variational methods and methods of weighted residuals [Finlayson, 1972]. Variational principles are used in the Rayleigh–Ritz method. Unfortunately, variational principles often cannot be found in some engineering problems, particularly when the differential equations are not self-adjoint. Weighted residuals are applied in the methods of Galerkin, least squares, and collocation. The method of weighted residuals utilizes a concept of orthogonal projection of a residual of a differential equation onto a subspace spanned by certain weighting functions. In the finite element method, we may use either variational principles when they exist, or weighted residuals through approximations. In finite element applications to fluid dynamics, the Galerkin method is often considered the most convenient tool for formulating finite element models since it requires no variational principles. The least squares method requires higher order interpolation functions in general, even if the physical behavior may be adequately described by linear or lower order functions. For these reasons, our discussions in this book are centered around the Galerkin method, although the finite element formulations via the methods of variational principles and least squares are demonstrated to a limited extent.

In the following sections of this chapter, we discuss basic mathematical preliminaries and notations. The reader is assumed to have been exposed to the vector analysis and matrix algebra. Tensor equations or index notations are used throughout the text, although no extensive knowledge of tensor algebra is required of the reader. All necessary mathematical preliminaries are presented in sufficient detail for the benefit of the beginner and for those who may require review. Brief discussions of functional analysis including Sobolev spaces, which is essential in error estimates and convergence, are also presented. Subsequently, we discuss the concepts of variational principles and weighted residuals as used in the classical approximate methods of analysis such as the Rayleigh–Ritz and Galerkin methods, respectively. The relationships of these classical concepts with the finite element theory are clarified. At the end of this chapter, simple example problems are solved to demonstrate the basic idea of finite element approximation for the beginner; more general discussions of finite element analysis are taken up in subsequent chapters.

1-2 MATHEMATICAL PRELIMINARIES

1-2-1 Vector and Index Notations

Some of the basic relations in mechanics may be written conveniently in vector notations. Let us begin with commonly used vector expressions in cartesian coordinates. Consider the vectors $\mathbf{A} = A_i\mathbf{i}_i$ and $\mathbf{B} = B_i\mathbf{i}_i$ with an angle between them being α. Here A_i and B_i are the components of the vectors $\mathbf{A}$ and $\mathbf{B}$, respectively, and $\mathbf{i}_i$ are the unit vectors with $i = 1, 2, 3$. The dot product and cross product are given by

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \alpha = A_i\mathbf{i}_i \cdot B_j\mathbf{i}_j = A_iB_j\delta_{ij} = A_iB_i \tag{1-1a}$$

$$|\mathbf{A} \times \mathbf{B}| = AB \sin \alpha = |A_i\mathbf{i}_i \times B_j\mathbf{i}_j| = |A_iB_j\varepsilon_{ijk}\mathbf{i}_k| \tag{1-1b}$$

where the repeated indices imply summing; δ_{ij} is the Kronecker delta having the property $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$; and ε_{ijk} is the permutation symbol defined as

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1 \qquad \text{(clockwise permutation)}$$

$$\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1 \qquad \text{(counterclockwise permutation)}$$

with all other $\varepsilon_{ijk} = 0$ for any two or more indices being repeated. The permutation symbol and Kronecker delta are related by

$$\varepsilon_{ijk}\varepsilon_{mnp} = \delta_{im}\delta_{jn}\delta_{kp} + \delta_{in}\delta_{jp}\delta_{km} + \delta_{ip}\delta_{jm}\delta_{kn} - \delta_{im}\delta_{jp}\delta_{nk} - \delta_{in}\delta_{jm}\delta_{pk} - \delta_{ip}\delta_{jn}\delta_{mp} \tag{1-1c}$$

The continued products are

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = A_iB_jC_k\varepsilon_{ijk} \tag{1-1d}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} = A_iB_jC_k\varepsilon_{jkl}\varepsilon_{ilm}\mathbf{i}_m = A_i(C_iB_j - B_iC_j)\mathbf{i}_j \tag{1-1e}$$

We introduce the vector differential operator, or simply del operator, ∇, defined as

$$\nabla = \mathbf{i}_i \frac{\partial}{\partial x_i} \tag{1-2}$$

where $\mathbf{i}_i$ are the components of the unit vector. Then

$$\nabla E = \mathbf{i}_i \frac{\partial E}{\partial x_i} = E_{,i}\mathbf{i}_i$$

where E is a scalar and the comma denotes the partial derivative. The divergence and curl are given by

$$\text{div } \mathbf{A} = \nabla \cdot \mathbf{A} = \frac{\partial A_i}{\partial x_i} = A_{i,i} \tag{1-3}$$

$$\text{curl } \mathbf{A} = \nabla \times \mathbf{A} = \varepsilon_{ijk} A_{j,i} \mathbf{i}_k \tag{1-4}$$

In view of (1-1) through (1-4) we can show that

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} \tag{1-5}$$

Let us consider the velocity vector $\mathbf{V} = V_i \mathbf{i}_i$. From (1-5), we may write

$$\nabla(\mathbf{V} \cdot \mathbf{V}) = 2\mathbf{V} \times (\nabla \times \mathbf{V}) + 2(\mathbf{V} \cdot \nabla)\mathbf{V} \tag{1-6}$$

in which

$$\nabla \times \mathbf{V} = \varepsilon_{ijk} V_{j,i} \mathbf{i}_k = \omega_k \mathbf{i}_k = \boldsymbol{\omega} \tag{1-7}$$

where ω_k are the components of vorticity vector

$$\boldsymbol{\omega} = \omega_1 \mathbf{i}_1 + \omega_2 \mathbf{i}_2 + \omega_3 \mathbf{i}_3 = (V_{3,2} - V_{2,3})\mathbf{i}_1 + (V_{1,3} - V_{3,1})\mathbf{i}_2 + (V_{2,1} - V_{1,2})\mathbf{i}_3 \tag{1-8}$$

For a two-dimensional problem, we have

$$\omega_3 = V_{2,1} - V_{1,2} \tag{1-9}$$

The acceleration vector may be written in the form

$$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} \tag{1-10a}$$

Substituting the relationship (1-6) in (1-10a) yields

$$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t} + \nabla\left(\frac{V^2}{2}\right) - \mathbf{V} \times \boldsymbol{\omega} \tag{1-10b}$$

in which $V^2 = \mathbf{V} \cdot \mathbf{V} = V_j V_j$. Using index notion, we have

$$a_k = \dot{V}_k + (\tfrac{1}{2} V_j V_j)_{,k} - \varepsilon_{ijk} V_i \omega_j$$

or

$$a_k = \dot{V}_k + (\tfrac{1}{2} V_j V_j)_{,k} - \varepsilon_{ijk} \varepsilon_{mnj} V_i V_{n,m} \tag{1-10c}$$

We have shown here that all vector equations may be written in terms of cartesian components of the vectors using index notations. The components a_i of the acceleration vector **a** may be referred to as a tensor of the first order. The scalar is the zero order tensor. The second order tensor, such as the stress tensor σ_{ij} or σ^{ij}, has two indices whereas the tensor of material constants is of fourth order E_{ijkl} or E^{ijkl}. It is seen that the number of indices determine the order of a tensor. If a vector or tensor is referred to a cartesian basis, we refer to their components as cartesian. For curvilinear coordinates and/or nonorthogonal coordinates, the resulting tensor equations are noncartesian leading to the covariant and contravariant components of a vector. Details of this subject are discussed in Sec. 4-1 [see Sokolnikoff, 1964].

1-2-2 Matrix and Index Notations

In continuum mechanics, many of the physical laws are expressed by differential equations which may then be transformed into tensor equations in a local or global form. The earlier development of the finite element analysis was made using the matrix equations because they appeared straightforward and convenient when dealing with linear problems. However, as many nonlinear problems were considered, the inadequacy of matrix notations became apparent. The power of tensor analysis in mechanics in general is well known [see Sec. 4-1; also, Sokolnikoff, 1967].

Consider a matrix equation of the form

$$\mathbf{A}\mathbf{u} = \mathbf{f} \tag{1-11}$$

where **A** represents a matrix of the size $m \times n$. A matrix with $m \neq n$ is called the rectangular matrix whereas that with $m = n$ is referred to as the square matrix. The matrix with $m = 1$ and $n > 1$ is called the row matrix whereas that with $m > 1$ and $n = 1$, the column matrix. Suppose that **A**, **u**, and **f** are of $m \times m$, $m \times 1$, and $m \times 1$, respectively, and we write (1-11) in the form

$$\begin{bmatrix} A_{11}A_{12}\cdots A_{1m} \\ A_{21}A_{22}\cdots A_{2m} \\ \vdots \quad\quad \vdots \\ A_{m1}A_{m2}\cdots A_{mm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix} \tag{1-12}$$

If we premultiply both sides of (1-11) by $\mathbf{u}^T$ with T denoting a transpose, then

$$\underset{1\times m}{\mathbf{u}^T}\;\underset{m\times m}{\mathbf{A}}\;\underset{m\times 1}{\mathbf{u}} = \underset{1\times m}{\mathbf{u}^T}\;\underset{m\times 1}{\mathbf{f}} \tag{1-13}$$

It is seen that (1-13) becomes a matrix of size 1×1, a scalar quantity, an invariant, or a form of energy which has a physical significance in continuum mechanics. Had (1-12) been postmultiplied by $\mathbf{u}^T$, the resulting matrix would be of $m \times m$

$$\underset{m\times m}{\mathbf{A}}\;\underset{m\times 1}{\mathbf{u}}\;\underset{1\times m}{\mathbf{u}^T} = \underset{m\times 1}{\mathbf{f}}\;\underset{1\times m}{\mathbf{u}^T} \tag{1-14}$$

If we choose to use index notations instead of matrix notations, we write (1-11) in the form

$$A_{ij}u_j = f_i \tag{1-15a}$$

or

$$u_jA_{ij} = f_i \tag{1-15b}$$

The repeated indices imply summing according to the ranges of indices. A repeated index is often called a dummy index and all indices not repeated are referred to as free indices. In the case of (1-15), i is free index and j is dummy index. If there is only one free index, then the number of equations is equal to the maximum range of the free index. If there is more than one free index, then the total number of equations is equal to the products of the maximum ranges of all free indices.

Suppose that in (1-15), $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$. Then there result m equations obtained by summing the products of the jth columns of A_{ij} and corresponding jth rows of u_j. It is obvious that the free indices of both sides of Eq. (1-15) must match. If $i,j = 1,2,\ldots,m$ in (1-15), the expanded form of (1-15) becomes

$$\begin{aligned} &A_{11}u_1 + A_{12}u_2 + \cdots + A_{1m} = f_1 \\ &A_{21}u_1 + A_{22}u_2 + \cdots + A_{2m} = f_2 \\ &\vdots \\ &A_{m1}u_1 + A_{m2}u_2 + \cdots + A_{mm} = f_m \end{aligned} \tag{1-16}$$

These equations are identical to the matrix form (1-12). If index notations are used for (1-13), we write

$$A_{ij}u_ju_i = f_iu_i \tag{1-17}$$

Note that the choice index i on u_i is equivalent to premultiplication of $\mathbf{u}^T$ in (1-13) and the index i is also repeated here. This will leave no free index and thus expansion of (1-17) results in a single equation

$$\begin{aligned} &A_{11}u_1u_1 + A_{12}u_2u_1 + \cdots + A_{1m}u_mu_1 \\ &+ A_{21}u_1u_2 + A_{22}u_2u_2 + \cdots + A_{2m}u_mu_2 \\ &+ \cdots \\ &+ A_{m1}u_1u_m + A_{m2}u_2u_m + \cdots + A_{mm}u_mu_m \\ &= f_1u_1 + f_2u_2 + \cdots + f_mu_m \qquad (m \text{ not to be summed}) \end{aligned} \tag{1-18}$$

which is identical to the expansion of the matrix equation (1-13). Likewise, Eq. (1-14) with index notation takes the form

$$A_{ij}u_ju_k = f_iu_k \tag{1-19}$$

It is clear that the free indices are i and k on both sides and thus the total number of equations is equal to $m \times m$. The expansion of (1-19) is of the form

$$B_{ik} = C_{ik}$$

or

$$\begin{bmatrix} B_{11}B_{12}\cdots B_{1m} \\ B_{21}B_{22}\cdots B_{2m} \\ \vdots \quad \vdots \quad\quad \vdots \\ B_{m1}B_{m2}\cdots B_{mm} \end{bmatrix} = \begin{bmatrix} C_{11}C_{12}\cdots C_{1m} \\ C_{21}C_{22}\cdots C_{2m} \\ \vdots \quad \vdots \quad\quad \vdots \\ C_{m1}C_{m2}\cdots C_{mm} \end{bmatrix}$$

where

$$B_{11} = A_{11}u_1u_1 + A_{12}u_2u_1 + \cdots + A_{1m}u_mu_1 = C_{11} = f_1u_1$$
$$B_{12} = A_{11}u_1u_2 + A_{12}u_2u_2 + \cdots + A_{1m}u_mu_2 = C_{12} = f_1u_2$$
$$\vdots$$
$$B_{mm} = A_{m1}u_1u_m + A_{m2}u_2u_m + \cdots + A_{mm}u_mu_m = C_{mm} = f_mu_m$$

(m not to be summed)

The results shown above are identical to those obtained by expanding the matrix equation (1-14).

Consider now a special case of a matrix equation of the form

$$\underset{6\times 3}{\mathbf{B}^T}\ \underset{3\times 3}{\mathbf{E}}\ \underset{3\times 6}{\mathbf{B}}\ \underset{6\times 1}{\mathbf{u}} = \underset{6\times 1}{\mathbf{F}} \tag{1-20}$$

The corresponding index equation may be obtained by introducing indices $\alpha, \beta = 1, 2, \ldots, 6$ and $i, j = 1, 2, 3$.

$$B_{i\alpha}E_{ij}B_{j\beta}u_\beta = E_{ij}B_{i\alpha}B_{j\beta}u_\beta = F_\alpha \tag{1-21}$$

with the free index α on both sides. Expanding (1-21) yields

$$\begin{aligned} &E_{11}B_{11}B_{11}u_1 + E_{11}B_{11}B_{12}u_2 + \cdots + E_{11}B_{11}B_{16}u_6 \\ &+ E_{12}B_{11}B_{21}u_1 + E_{12}B_{11}B_{22}u_2 + \cdots + E_{12}B_{11}B_{26}u_6 \\ &+ E_{13}B_{11}B_{31}u_1 + E_{13}B_{11}B_{32}u_2 + \cdots + E_{13}B_{11}B_{36}u_6 \\ &\quad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\quad \vdots \qquad\qquad \vdots \\ &+ E_{33}B_{31}B_{31}u_1 + E_{33}B_{31}B_{32}u_2 + \cdots + E_{33}B_{31}B_{36}u_6 = F_1 \\ &\quad\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \end{aligned}$$

To obtain the first equation from the matrix equation (1-20), one must write out completely all components of each matrix, but these matrix multiplications may become quite cumbersome.

Let us now consider a fourth order symmetric tensor E_{pqrs} with $p, q, r, s = 1, 2$

$$E_{pqrs} = \begin{bmatrix} E_{1111} & E_{1122} & 0 \\ E_{2211} & E_{2222} & 0 \\ 0 & 0 & E_{1212} \end{bmatrix}$$

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