Mechanical and Electrical Vibrations

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Preface

This is an account of vibrations in which the illustrative material is taken from many branches of physics and engineering. Vibrations is a subject in which a student can apply, compare and relate many of the fundamental ideas of mechanics and of electricity by means of mathematically-based analogies. Such an integrating study has obvious value at a time when scientific and technical knowledge is increasing with almost explosive rapidity, when a thorough understanding of fundamental physical concepts and of mathematics is essential.

The book describes certain mathematical ideas and explains their application to physical systems: not mathematics-by-itself but mathematics-with-physics. Such an approach is particularly suited to those physics and engineering students whose mathematics is weak, although all may benefit from it. For engineers in particular, the acquisition of skill in, say, algebra and analysis, is less important than in being able to formulate physical problems in mathematical language: following this, the solution of hitherto intractable problems becomes possible by computers. This book is therefore offered as a contribution to the art of formulating problems in mathematical terms and in interpreting mathematical formulae in physical terms.

The mathematics is deliberately limited so that undergraduate students will not be unduly harassed; elementary differential equations and complex numbers are the main tools needed. The book will, however, guide the student towards more advanced topics such as Laplace transforms, matrices, and numerical methods. Some sections of the book are at postgraduate level, e.g. Ch. 7 on Lagrange's equations, and may be omitted on a first reading.

Of particular interest is the account of transducer theory in Ch. 10

for, so far as is known, no earlier textbook contains so complete a treatment of the basic theory.

The book is partly based on a lecture course on Vibrations and Waves given by the writer some years ago to physics students at Queen Mary College. It has not proved possible to include an account of wave-motion in this book but a second volume covering this subject will be written. Thanks are due to the physics staff at Queen Mary College who suggested that the book should be written, to colleagues at Imperial College who have read and criticized various portions of the manuscript – notably Messrs Birss, Mautner, Mayne, Michaelson and Prigmore – and to my wife who has typed the manuscript.

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CHAPTER 1

Introduction

The study of vibrations, besides being important for its practical applications, is also useful as a link between the various branches of physics in which vibrations occur. Examination of the analogies between electrical and mechanical vibrating systems can aid the understanding of both electricity and mechanics, and this aspect of the subject, particularly, is developed in this book. Technically, vibrations and waves are of immense importance: the mechanical engineer is often engaged in suppressing them because vibrations in machinery can cause discomfort (e.g. in motor-cars) and even danger (e.g. in unbalanced rotating machines), while the electrical engineer exploits them to pass information and energy between distant places.

The simplest natural vibrating systems have two attributes: inertia and stiffness-about-an-equilibrium-position. To a first approximation, the force tending to restore the system to equilibrium is often proportional to the displacement and then simple harmonic motion occurs. This kind of motion, which can occur in systems which are mechanical, hydraulic, acoustical, optical or electrical, partly accounts for the immense importance of the sinusoidal function in physics. If a natural vibrating system is set into forced motion, this too will be approximately sinusoidal if the applied forces are sinusoidal functions of time. (Here the words 'force' and 'motion' are used in a general sense which includes the electrical and other cases.) Often, forced motions due to a variety of forces applied simultaneously seem to have some independence in the sense that the total forced motion is approximately the sum of the motions due to the forces applied one at a time. This leads to an abstraction from reality, an idealization, known as the 'principle of superposition', which puts a wealth of mathematics at our service; where it does not apply, the mathematical studies are much more difficult. The principle of superposition further enhances the importance of the sinusoidal function because it is possible to analyse a wide range of time-varying applied forces into sinusoidal components by Fourier's series or integral. The response to each component can be found separately, which is comparatively easy, and then superimposed by addition or integration.

No very great complication is introduced if the vibrations gradually decrease in amplitude because of energy losses from the system, provided that the dissipative forces can be assumed to be proportional to the velocities. (This is so for viscous friction and, via an analogy, for electrical resistance but not for Coulomb friction unfortunately.) With dissipative forces proportional to velocity, opposing the motion, the amplitude decreases exponentially with time.

The sine, cosine and exponential functions are a unique class whose derivatives and integrals are all members of the same class. This simple mathematical fact is related to the physical fact that an oscillatory driving force so often produces an oscillatory response. In many cases, this fact also makes the solution of the equations of motion comparatively simple because it is easy to guess a suitable mathematical form for the solution!

The plan of this book is straightforward. In Ch. 2, simple vibrations are studied and methods of calculating their natural frequency and decay rates are given. The 'quality factor', Q, is used to describe the decay of free vibrations rather than the older 'logarithmic decrement' but the relation between them is derived.

In Ch. 3, the analogies between various physical systems are explained so that the same mathematics can be applied to a variety of cases: this is one of the main features of the book. There are two analogies between electrical and mechanical systems. In the better known one, the energy stored in the magnetic field of a coil is regarded as similar to the kinetic energy of a mass, and electrostatic energy in a capacitor is regarded as similar to potential energy. In the other analogy, which is actually simpler to use although both are of equal status, these associations are reversed.

Forced vibrations are studied in Ch. 4 where the complex-number method of finding the steady-state (sinusoidal) forced vibrations of a linear system is introduced. With this is associated the concept of 'impedance' used especially by electrical engineers but applicable equally to mechanical vibrating systems. Anti-vibration mountings and electrical oscillators provide some of the illustrative material, and the phenomenon of 'resonance' is also described in detail.

Chapter 5 deals with a variety of ideas, particularly those associated with the principle of superposition. This book only deals with the vibrations of 'lumped-constant systems', as distinct from 'distributed-constant systems', and what this means is described at the beginning of Ch. 5. Linear systems are also distinguished from non-linear ones.

In Ch. 6, more complicated vibrating systems – those with more than one mode of free vibration – are introduced. Setting-up the differential equations which describe such systems is the first topic, and by exploiting the analogies of Ch. 3, nine related physical systems are produced as examples. Then follows an account of the solution of the differential equations, which begins with an example and concludes with some remarks about the general case, including the production of numerical results. As it may often be of interest merely to know how many modes of vibration a system has, a rule is given for ascertaining this without solving the differential equations. In the last section, the physical behaviour of a particular mass-spring system (one of the nine systems mentioned above as having related mathematical descriptions) is examined in detail and the idea of 'natural coordinates' is introduced. Second-year science and engineering undergraduates might stop at this point, although some electrical students may be ready to look ahead to Ch. 8.

Chapter 7 introduces more advanced work. The method of analytical dynamics, as developed by Lagrange, provides a powerful tool for the analysis of complex mechanical systems. Here, Lagrange's equations – in a form specially suited to deal with small vibrations about a static equilibrium position – are derived from Newton's laws of motion. It is then possible to generalize many of the results of the earlier Chapters and to see, for example, how it is that linear differential equations with constant coefficients are the appropriate mathematical description for such a wide variety of mechanical vibrators. Lagrange's equations also provide a simple error-free method for setting-up the equations for complex physical systems. The transformation to normal coordinates, for a dissipation-free system, and the orthogonal nature of the normal modes of vibration are discussed in geometrical terms. The free vibrations of a particle confined to the vicinity of a fixed point by a field of force provide a simple example, following which both the free and forced transverse-vibrations of a loaded string are described. A train of goods wagons coupled by springs, with no friction, executing longitudinal vibrations, is an analogue of the loaded string and there are electrical analogues too. 'Rayleigh's principle' by which the natural frequencies and vibration shapes can be estimated without directly solving the differential equations, the effect of dissipation on the normal modes, and the applications of Lagrange's equations to electric circuits, complete this chapter.

Chapter 8, on the concepts of admittance and impedance, is almost independent of Ch. 7. Those concerned with electrical systems will already know how important these concepts are, but it must be emphasized that the enormous wealth of electric circuit theory is available, via the electromechanical analogies, for application to mechanical systems. Mechanical engineers should not neglect this and there is an analysis of the dynamic vibration absorber which should be of interest. (On the other hand, electrical engineers should not neglect the methods of analytical dynamics introduced in the previous chapter.)

The early chapters have dealt with systems vibrating about a position of stable equilibrium, but Ch. 9 introduces vibration about a state of motion. Gyroscopes and mass-spring systems on a rotating table are used as examples and have some surprising physical properties – surprising, that is, to those who previously have only studied vibrations about a stable equilibrium position. The stabilization of ships in rough seas by a gyroscope is one of the examples and serves to illustrate a few remarks about 'feedback' and 'control systems'. To provide electrical analogues of these mechanical systems, a new theoretical electrical element called the 'gyrator' is defined and its practical physical realization, so far as it has been achieved, is described. It proves convenient at this stage to introduce the 'ideal transformer' as the electric analogue of an idealized mechanical lever or pair of gear-wheels, ready for use in Ch. 10.

This final chapter provides something of a climax in analysing systems containing both mechanical and electrical elements, namely 'electromechanical transducers'. These include both moving-coil and electrostatic loudspeakers and microphones. Gyrators appear in some of the equivalent circuits. Two different idealized transducers are defined, and these, together with gyrators, transformers and gyroscopic couplers form an interesting class of related devices.

CHAPTER 2

Simple Free Vibrations

SIMPLE HARMONIC MOTION

Simple harmonic motion is most easily defined as the projection of circular motion of constant angular speed ω on to a straight line in the plane of the motion. If the point P in Fig. 1 moves in this way, the projection of the rotating radius CP $= r$ on to a line AB is

$$OY = y = r \, . \sin(\omega t + \phi) \tag{2.1}$$

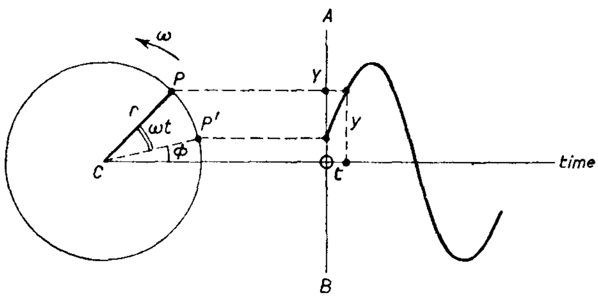


FIG. 1 Simple harmonic motion of Y along AB.

and the point Y is said to execute simple harmonic motion along AB. In the diagram, CO is a convenient reference direction perpendicular to AB, and O defines the central position of the motion of the point Y on AB. P′ is the position of P at time $t = 0$ and if $\angle$OCP′ $= \phi$ then $\angle$OCP $= \omega t + \phi$. A graph of y versus t is shown at the right of the diagram. r is called the AMPLITUDE of the motion, ϕ is called the PHASE ANGLE and ω the ANGULAR FREQUENCY or PULSATANCE.

Alternatively, simple harmonic motion can be defined as the motion of a point Y along a straight line AB subject to the condition that its acceleration is always directed towards a fixed point O on that line, and has a magnitude proportional to its displacement y from O. If the constant of proportionality is ω^2, this gives the differential equation

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \tag{2.2}$$

whose solution is known to be

$$y = a\sin\omega t + b\cos\omega t \tag{2.3}$$

where a and b are determined by the initial conditions, namely the velocity and displacement at $t = 0$.

The relationship between Eqns. 2.1 and 2.3 is that

$$a^2 + b^2 = r^2 \quad \text{and} \quad \tan^{-1}\frac{b}{a} = \phi$$

The PERIOD OF OSCILLATION τ is the time taken for the point P to go once round the circle, hence

$$\tau = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{(\textit{acceleration at unit displacement})}} \tag{2.4}$$

The FREQUENCY f is the reciprocal of the period τ and equals the number of revolutions of P per unit time.

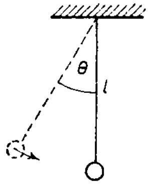
The differential equation (2.2) arises in many idealized physical situations of which a simple undamped pendulum, a mass-spring vibrator with one degree of freedom, and an L–C circuit are illustrated in Fig. 2. In the first two of these cases, the period τ can be obtained by applying Eqn. 2.4. In the electrical case, the equation

$$L\frac{di}{dt} + \frac{q}{C} = 0$$

in which i is the current in the inductor and q the charge on the capacitor, expresses the fact that the sum of the voltage drops round the loop is zero. Putting $i = dq/dt$ and dividing through by L, gives

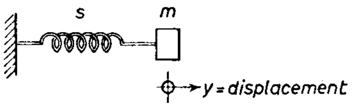
$$\frac{d^2q}{dt^2} + \frac{q}{LC} = 0$$

which is of the same form as Eqn. 2.2 so that $\omega^2 = 1/(LC)$.



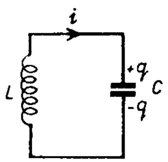
SIMPLE PENDULUM

Acceleration due to gravity $= g$
For small displacements,
$\tau = 2\pi\sqrt{(l/g)}$ and
$\omega^2 = g/l$



MASS-SPRING SYSTEM

mass $= m$
spring stiffness
$= s$
$=$ force/displacement
$\tau = 2\pi\sqrt{(m/s)}$
$\omega^2 = s/m$



L–C CIRCUIT

$\tau = 2\pi\sqrt{(LC)}$
$\omega^2 = (LC)^{-1}$

FIG. 2 Simple harmonic vibrators.

DAMPED VIBRATIONS

Mechanical, electrical and acoustic vibrations do not go on for ever at constant amplitude* because of dissipation of energy by mechanical friction, electrical resistance, radiation, etc. This is known as DAMPING.

The simplest mechanical case to analyse is one where a force proportional to velocity opposes the motion: for the mass-spring system of Fig. 3a, such a force $r(dy/dt)$, where $r =$ constant, is provided by the viscous oil damper (as used on a motor car). The

* Internal vibrations of isolated molecules may seem to provide an exception but a collision with another molecule or a photon will eventually occur in practice and the energy will be changed.

differential equation stating that the *mass* × *acceleration* = *sum of forces* is

$$m\frac{d^2y}{dt^2}+r\frac{dy}{dt}+sy = 0 \tag{2.5}$$

and the solution is

$$y = A\epsilon^{-(r/2m)t}.\cos(\omega_n t+\phi) \tag{2.6}$$

where
$$\omega_n^2 = \frac{s}{m}-\frac{r^2}{4m^2} = \omega_0^2-\frac{r^2}{4m^2} \tag{2.7}$$

so that the 'natural' angular frequency ω_n (i.e. that belonging to free vibration) is reduced when damping is present. ω_0 is the value of ω_n when there is no damping.

In Eqn. 2.6, A and ϕ can be determined from the initial values of y and dy/dt. The amplitude $A\epsilon^{-(r/2m)t}$ decreases exponentially by the factor ϵ in a time interval $2m/r$: this interval is known as the TIME-CONSTANT T of the decay.

Some examples of idealized damped vibrators which all obey the same type of differential equation as 2.5 are given in Fig. 3. (In 3d, the equation expresses the fact that the sum of the currents at node A is zero.) Moving-coil galvanometers also obey a differential equation in this class, the ballistic ones having the least possible damping. It is therefore convenient to decide on a standard form of the equation, and the following is chosen:

$$\frac{d^2y}{dt^2}+\frac{\omega_0}{Q}\frac{dy}{dt}+\omega_0^2 y = 0 \tag{2.8}$$

Here, the magnitude of the new dimensionless parameter Q, called the Q-FACTOR, represents the quality of the vibrator in that if Q is large, the damping-term is small. Taking the systems of Fig. 3 in turn, the values of Q and ω_0 are, respectively

(*a*)	$Q = \omega_0 m/r$	$\omega_0 = \sqrt{(s/m)}$
(*b*)	$\omega_0 J/r$	$\sqrt{(s'/J)}$
(*c*)	$\omega_0 L/R$	$1/\sqrt{(LC)}$
(*d*)	$R/(\omega_0 L)$	$1/\sqrt{(LC)}$

In each case, the natural frequency $\omega_n/(2\pi)$ of the vibrator is (from Eqn. 2.7)

$$\omega_n^2 = \omega_0^2\left(1-\frac{1}{4Q^2}\right) \tag{2.9}$$

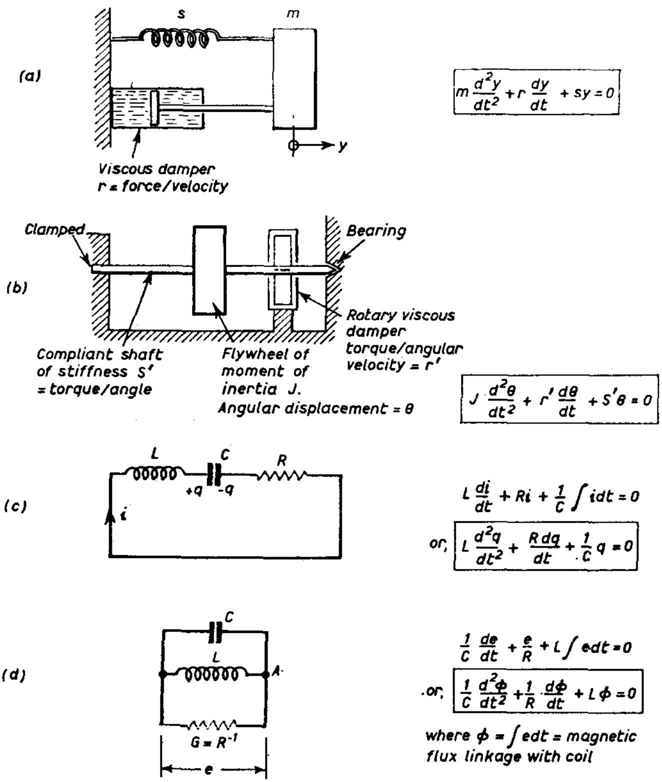


FIG. 3 Simple damped vibrators.

and the time-constant of the decay is

$$T = 2Q/\omega_0 \tag{2.10}$$

The number N of cycles of the vibration needed for the amplitude to fall to $1/\epsilon$th of its value is the number of cycles in a time interval equal to T, thus

$$N = \frac{T}{\tau} = \frac{2Q}{\omega_0}\frac{\omega_n}{2\pi} = \frac{Q}{\pi}\sqrt{\left(1-\frac{1}{4Q^2}\right)}$$

If Q is greater than (say) 4, the value of N is very close to Q/π.

A satisfactory definition of the Q-factor of the vibrator, is found by considering Eqn. 2.8.

$$\left.\begin{aligned} &\text{If} \quad \frac{\text{coefficient of the } y\text{-term}}{\text{coefficient of } d^2y/dt^2} = \omega_0^2 \\ &\text{then,} \quad Q = \omega_0 \frac{(\text{coefficient of } d^2y/dt^2)}{(\text{coefficient of } dy/dt)} \end{aligned}\right\} \qquad (2.11)$$

This somewhat formal definition, suggested by Morris[1], is free from the objections associated with some of the definitions commonly used by radio engineers. One such definition is

$$\left.\begin{aligned} Q &= \frac{2\pi}{\textit{the fraction of the energy dissipated per cycle}} \\ &= 2\pi\left(\frac{\textit{peak value of energy stored}}{\textit{energy dissipated per cycle}}\right) \end{aligned}\right\} \qquad (2.12)$$

For a mechanical resonator, the peak energy referred to may be either potential or kinetic according to whether it is calculated at the instant of maximum displacement or maximum velocity. Similarly, for the electrical case, the peak value of either the electrostatic or magnetic energy may be taken. Thus, for the electrical case of Fig. 3c, if the R.M.S. current is I and the peak current is I_m,

$$Q = 2\pi\frac{\frac{1}{2}LI_m^2}{RI^2\tau} = 2\pi\frac{\frac{1}{2}LI_m^2}{\frac{1}{2}RI_m^2\tau} = \frac{2\pi L}{R\tau} = \frac{\omega L}{R}$$

This calculation is only meaningful if the damping is slight because no distinction has been made between ω_n and ω_0, nor between the peak values at the beginning and end of a cycle, nor of the departure from sinusoidal conditions due to damping. Thus Eqn. 2.12, while useful in providing a physical interpretation of Q, does not give a clear and unambiguous definition of it. Equations 2.11 are to be preferred because they apply to both light and heavy damping.

Other measures of the quality of a vibrator are the DAMPING FACTOR δ, which is the reciprocal of the Q-value, and the LOGARITHMIC DECREMENT γ. The latter has long been associated with ballistic galvanometers in physics textbooks: the ratio of successive swings on opposite sides of the rest position is measured and its natural logarithm is defined as the logarithmic decrement γ. The ratio is found from values of $\epsilon^{-t/T}$ taken at two times $\frac{1}{2}\tau$ apart, hence,

$$\begin{aligned} \gamma = \log_\epsilon(\text{ratio}) &= \tfrac{1}{2}\tau/T \\ &= \frac{\textit{half period of free vibration}}{\textit{time constant of decay}} \\ &= \frac{\pi/\omega}{2Q/\omega} \\ &= \frac{\pi}{2Q} \end{aligned} \tag{2.13}$$

No distinction is made between ω_n and ω_0 as light damping can be assumed.

Whatever method is adopted for solving the differential equation (2.8), it will always be necessary to find the roots of the so-called CHARACTERISTIC QUADRATIC EQUATION

$$p^2 + \frac{\omega_0}{Q}p + \omega_0^2 = 0 \tag{2.14}$$

These roots determine the natural angular frequency ω_n and time-constant T of decay:

$$\begin{aligned} p &= -\frac{\omega_0}{2Q} \pm \sqrt{\left(\frac{\omega_0^2}{4Q^2} - \omega_0^2\right)} \\ &= -\frac{\omega_0}{2Q} \pm j\omega_0\sqrt{\left(1 - \frac{1}{4Q^2}\right)}; \qquad j = \sqrt{(-1)} \\ &= -\frac{1}{T} \pm j\omega_n \end{aligned} \tag{2.15}$$