THE CRESCENT DICTIONARY OF MATHEMATICS

by William Karush

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INTRODUCTION

The subject matter of the 1422 entries in this dictionary covers two general categories. First, a detailed treatment is provided of the following standard high school and college mathematics material: arithmetic; elementary, intermediate, and college algebra; plane and solid geometry; plane and spherical trigonometry; plane and solid analytic geometry; differential and integral calculus. Second, a wide selection of items is provided from more advanced mathematics, including the following fields: logic and fundamental concepts; theory of equations, theory of numbers, and modern higher algebra; advanced calculus; geometry and topology; probability and statistics; recent areas such as computer sciences, information theory, operations research, and so on.

The explanations of entries range from brief definitions of a few lines to expository discussions of various lengths. Several goals motivated the treatment. The dictionary was to satisfy the immediate technical needs of the user. Mathematics was to be presented as a unified, ever-growing, and stimulating body of knowledge. The reader who was a nonspecialist in mathematics was to be served in advanced mathematics and in new fields (such as computer sciences) that were significant for the future. In terms of its broader objectives, the present book may be viewed as an outline or compendium of mathematical ideas correlated with a large collection of specialized items. There are concern and activity in the educational field with regard to revitalizing and modernizing scientific training and creating a better understanding of science; it is hoped that this volume will contribute to these aims in mathematics.

The dictionary is intended for several audiences. First, it is designed to meet the educational needs of the participants in high school and college mathematics—the student and the teacher. It serves as a convenient summary of the concepts, formulas, and techniques studied in courses; beyond this, it may be used as an avenue for the student to step into fresh realms of mathematical ideas. The dictionary is also intended for many professional workers in behavioral and social sciences, biological and physical sciences, and engineering; it aims to satisfy both their needs in techniques of mathematics as well as their interest in advanced concepts and recent developments. Finally, it should serve the general reader with an intellectual interest in mathematics.

Several practices were adopted to make the dictionary as useful as possible for the intended class of readers. Specific formulas are given for solving problems, and examples and diagrams are called upon frequently for clarification. Attention is confined largely to terms and definitions that are in common use and that are pertinent for the reader. An attempt is made to reduce to a minimum the need to use cross references to find a given term. Related information is often introduced to stimulate interest; this, for example, may be of a historical nature or it may connect the given term to other terms. With regard to the question of technical rigor, the following view was adopted: in the case of an entry for which a formal definition meeting the requirements of the professional mathematician was judged as being too technical, a balance was attempted between mathematical precision and an informal description meaningful to the reader; when it seemed helpful, both an informal and a formal description were given.

Various reference works were helpful in the preparation of this dictionary. Especially useful were the following two: (1) *The Development of Mathematics*, by Eric T. Bell, 2nd edition, McGraw-Hill, New York, 1945; (2) *Mathematics Dictionary*, edited by Glenn James and Robert C. James, 2nd edition, D. Van Nostrand Co., Princeton, New Jersey, 1959.

The author is happy to express his gratitude to his generous colleagues, Dr. Richard E. Bellman and Dr. Mario L. Juncosa, for their comments and advice in the writing of this book. He also wishes to thank Professor H. S. MacDonald Coxeter for his suggestions on a number of entries, and Professor Kenneth O. May for his general remarks on an initial version of the manuscript. His gratitude is extended to Mr. Oscar Tarcov, General Editor of the Crescent Dictionaries, and Miss Lee Deadrick of The Macmillan Company for the opportunity to write one of the dictionaries in this series, and for their solicitous cooperation in many matters connected with the publication of this book. Thanks are also expressed to the many others who in their individual ways helped to create whatever qualities this work may possess.

William Karush

A GUIDE TO THE USE OF THE DICTIONARY

All entries have been entered in strict alphabetical order.

Boldface type has been used throughout for all words and terms for which definitions are included. The cross references are planned to facilitate fuller exploration of a subject beyond an initial, specific question. Therefore, in addition to main entries, all cross references appear in boldface; a cross reference may either be designated as such, or it may occur as a word or phrase in a definition.

Italics have been used to emphasize certain words and phrases within the body of the definition.

Synonyms are indicated by the abbreviation (syn.)

References for further reading are indicated at the end of certain entries by the abbreviation (Ref.) together with the appropriate number as listed in the List of References beginning on page 293.

CONTENTS

Abacus (Syn.: Counting Board) A device of ancient origin used for recording numbers and calculating. It has had several forms during its history, but the most familiar one is a frame with several parallel rods and individual counters that are free to slide along the rods. One rod is for the units place, the next for the tens place, and so on; in carrying out a calculation the counters are moved back and forth on the rods. The abacus was widely used in Europe by merchants up to the latter half of the thirteenth century, when it began to yield to the superior arithmetic of the Hindu-Arab decimal number system. Ref. [19, 24, 56, Vol. 1].

Abscissa The first, or horizontal coordinate, x, of a pair (x, y) of **Cartesian coordinates in the plane**. The second, y, is the *ordinate*.

Absolute Convergence An infinite series $a_1 + a_2 + a_3 + \cdots$ *converges absolutely* in case the series

$$|a_1| + |a_2| + |a_3| + \cdots$$

of absolute values converges. Absolute convergence implies (ordinary) convergence of the given series. An example of an absolutely convergent series is:

$$1 - \tfrac{1}{2} + (\tfrac{1}{2})^2 - (\tfrac{1}{2})^3 + \cdots$$

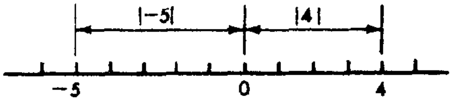
(the series of absolute values, $1 + \frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \cdots$, is a convergent geometric series). See **Conditional convergence**.

Absolute Inequality See **Unconditional inequality** (syn.).

Absolute Maximum (Minimum) Of a function, the largest (smallest) value. See **Function of real variable**.

Absolute Number See **Constant** (syn.).

Absolute Value (Syn.: Numerical value) The absolute value of a number, geometrically, is its distance from the zero point on an ordinary number scale (regardless of direction). For example, the absolute value of 4 is 4, of -5 is 5, and of 0 is 0. The



absolute value of a is denoted by $|a|$; for example,

$$|4| = 4, |-5| = 5, |0| = 0.$$

See **Linear coordinates**.

A formal definition of $|a|$ is: if a is positive or zero, then $|a| = a$; if a is negative, then $|a| = -a$. For example, $|-4| = -(-4)$, or 4.

The following properties of absolute value are important:

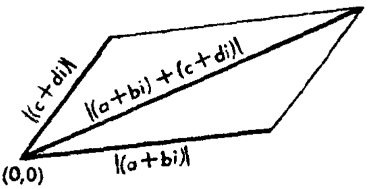
(1) $|a| \geq 0$;
(2) $|a \cdot b| = |a| \cdot |b|$;
(3) $|a + b| \leq |a| + |b|$.

As an example, verifying these, let $a = -2$ and $b = 7$; by **(2)**, $|-14| = |-2| \cdot |7|$, or $14 = 2 \cdot 7$; by **(3)**, $|(-2) + 7| \leq |-2| + |7|$, or $5 \leq 2 + 7$.

Absolute Value of Complex Number (Syn.: Magnitude; Modulus) For a **complex number** $a + bi$, the absolute value is the (real) number $\sqrt{a^2 + b^2}$; it is the distance of the complex number from the origin, when the complex number is represented as the point with rectangular coordinates (a, b). For example, the absolute value of $-2 + 3i$ equals $\sqrt{4 + 9}$, or $\sqrt{13}$. The absolute value is denoted by $|a + bi|$. The following properties are important:

(1) $|a + bi| \geq 0$;
(2) $|(a + bi)(c + di)| = |a + bi| \cdot |c + di|$;
(3) $|(a + bi) + (c + di)| \leq |a + bi| + |c + di|$

(this last expresses the "triangle inequality," namely, that the length of



a side of a triangle is not greater than the sum of the lengths of the other two sides).

Absolute Value of Vector The length of a **vector V**; it is often denoted by $|\mathbf{V}|$.

Abstract Mathematics Mathematics as knowledge apart from its meaning in terms of physical or concrete experience. Such experience may guide the development of a mathematical theory, and it may provide a helpful way to think about mathematical concepts; however, the conclusions of an abstract theory are expressed and deduced by logical means which are independent of "real" or "specific" interpretation. For example, it may be helpful to regard a *number* as a physical quantity, but the theory of real numbers can be developed as a deductive theory which requires no reference to this, or any other, specific meaning. See **Axiomatics, Deductive theory, Mathematical system, Mathematics**. Ref. [71].

Abstract Space In general, a formal mathematical system of a geometric-like nature. It is a **deductive theory** whose undefined terms and axioms, typically, "abstract" the features common to several more specific, or more familiar, systems; the latter serve as *models* of the abstract system. See **Hilbert space, Metric space, Vector space.**

Abundant Number (Syn.: Redundant number) See **Perfect number**.

Acceleration The rate of change of **velocity** with respect to time t. If the motion of a particle in the plane is given by $x = x(t), y = y(t)$, then the second derivatives $(d^2x)/(dt^2)$, $(d^2y)/(dt^2)$ give the components a_x, a_y of the acceleration along the x-axis and y-axis, respectively. The absolute magnitude of the acceleration is $\sqrt{a_x^2 + a_y^2}$; this equals the second derivative $(d^2s)/(dt^2)$ (where $s(t)$ is distance along the curve from a fixed point), except possibly for sign.

Accuracy Of an **approximate number**, a numerical measure of its closeness to the true value for which it stands. For example, 3.1416 is commonly used as an approximate value of π; it is said to be *accurate*, or correct, to four decimal places, meaning that the true value of π lies between 3.14155 and 3.14165. A decimal approximation, when all its digits are *significant*, is correct to the last decimal place shown, and the **error** is then no more than $\frac{1}{2}$ the unit of the last place.

Accuracy of Table The accuracy of the numerical entries in a table, such as a table of common logarithms or of the values of a trigonometric function. Nearly all entries in such tables are approximate, because the true functional values are typically unending decimals. See **Table of function.**

Acute Angle An **angle** of numerical measure less than 90 degrees.

Acute Triangle A **triangle** with each of its angles an acute angle.

Addend Any one of the individual constants of an expressed sum of constants. For example, in $2 + 3 + 5$, the addends are 2, 3, and 5.

Addition Addition of numbers is one of the fundamental operations of arithmetic. The result of combining two numbers a and b by addition is the *sum* of the numbers, and is denoted by $a + b$ (the symbol "+" was introduced about 1500). The sum of two whole numbers, say 3 and 5, is interpreted as the number of objects in the collection obtained by putting together a collection of 3 objects and a collection of 5 other objects. As the concept of number is extended to negative numbers, rational numbers (fractions), and, finally, the full set of real numbers, the meaning of addition is similarly extended. The rules of arithmetic provide ways of carrying out addition of numbers in various forms. **Algebraic addition**, for example, shows how to add signed numbers. See **Common denominator**, **Transfinite number**.

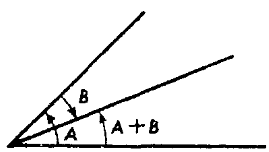
Addition of numbers is a binary operation—it combines two things (numbers), to produce a single thing. This operation has certain basic properties. One is the existence of the particular number 0 which leaves any number a unchanged under addition; that is, $a + 0 = a$ (0 is called the "identity element" of addition). Another property is that addition is associative; that is $(a + b) + c = a + (b + c)$. Also, addition is commutative; that is, $a + b = b + a$. The associative and commutative properties account, for example, for the fact that a sum of any number of terms can be checked by adding in the reverse order. Such general properties and others that account for certain aspects of the behavior of numbers, are treated in higher algebra in the study of the **field**.

Addition Formulas (trigonometry) Formulas which express a trigonometric function of the sum (or difference) of two angles in terms of the functions of the individual angles. The most commonly used are the following:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$
$$\cos(A + B) = \cos A \cos B - \cos A \sin B$$
$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$
$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$
$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

Addition of Algebraic Expressions In adding algebraic expressions, *similar terms* can be combined by adding their coefficients; for example $(3x^2 - 2xy + 3) + (y^2 + 3xy - 5)$ can be reduced to $3x^2 + xy + y^2 - 2$ by this means. See **Common denominator** for addition of algebraic fractions.

Addition of Angles The sum $A + B$ of two directed angles (rotations) A and B is an **angle** constructed as the rotation A followed by the rotation B (positive angles being counterclockwise and negative angles, clockwise). The measure of $A + B$ is the sum of the measures of A and B. For example, the sum of a 60° angle and



a 30° angle is a 90° angle; the sum of a 60° angle and a −30° angle is a 30° angle.

Addition of Complex Numbers See **Complex number**.

Addition of Sets See **Union of sets**.

Addition of Vectors (Syn.: Composition of vectors) When vectors $\mathbf{V}_1$, $\mathbf{V}_2$ are represented as directed line segments, their sum, or resultant,

$\mathbf{V}_1 + \mathbf{V}_2$ is the directed diagonal of the parallelogram whose adjacent sides are $\mathbf{V}_1$ and $\mathbf{V}_2$ (this is the parallelogram law). When plane vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ are represented by number pairs, as (x_1, y_1), (x_2, y_2), their sum is the vector $(x_1 + x_2, y_1 + y_2)$, obtained by adding like coordinates; this rule applies to vectors in a space of any number of dimensions.

Adjacent Two *angles* are adjacent if they share a common vertex and a common side, but do not overlap. Two *sides* of a triangle, or polygon, are adjacent in case they share a common vertex.

Aleph-Null, $\aleph_0$ In the theory of sets, the smallest infinite **cardinal number**; it is the cardinal number of the unending sequence of whole numbers $\{1, 2, 3, 4, \ldots\}$. ($\aleph$ is the first letter of the Hebrew alphabet.)

Algebra Ordinary algebra is the study of operations and relations among numbers through the use of **variables**, or literal symbols, such as "*a*," "*b*," "*x*," "*y*," etc., instead of just constants, such as "2," "$\frac{1}{6}$," etc. The use of variables gives algebra vastly greater scope than arithmetic, which is limited principally to constants. For example, in algebra, the distributive law of numbers is expressed by the formula "$a \cdot (x + y) = ax + ay$." In arithmetic, only specific instances could be cited, such as $3(5 + 8) = 3 \cdot 5 + 3 \cdot 8$, $2(1 + 6) = 2 \cdot 1 + 2 \cdot 6$; or the law might be stated as a verbal rule: "The product of a given number into a sum of two numbers is the product of the given number into the first plus its product into the second." Not only are symbolic formulas more compact than verbal rules, but also they permit manipulations for solving equations and deriving new relationships which would be extremely difficult or impossible without them.

The role of the Arabs in spreading algebra throughout Western Europe (beginning about 800 A.D.) is important in the history of the subject. However, present-day symbolism was not perfected in Europe until the sixteenth and seventeenth centuries. Algebra was eventually founded on a logical basis in the nineteenth century; here, the rules and formulas of ordinary algebra can be derived from a few initial axioms, much as the theorems of plane geometry are deduced from axioms and postulates. Many of the features of algebra follow from the axioms of the **field**.

As an elementary subject, algebra treats techniques for handling algebraic expressions, the solution of equations, and related topics. Interpreted more broadly, it is one of the major divisions of **mathematics**; it is the part which deals with "finite" processes, rather than "infinite" processes, such as are typical of the calculus. The description "higher algebra" is often applied to the advanced aspects of the subject, as distinguished from the more familiar, ordinary algebra. Examples of topics in higher algebra are *algebraic numbers, groups, fields, rings, number fields,* and the *algebra of matrices*. In a very general sense, an "algebra" is a mathematical system which expresses itself in variables and symbols for its entities, operations, and relations, and develops formal rules for the manipulation of its expressions.

Ref. [32, 34, 35, 45, 61].

Algebra, Boolean See **Boolean algebra**.

Algebra of Matrices The symbolic study of matrices, and operations and relations among them; it is a generalization of ordinary algebra. In this algebra a **matrix**, such as a second order matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is represented by a single letter, A, B, C, etc.; addition and multiplication are denoted by $A + B$ and $A \cdot B$ (or AB) as in ordinary algebra, and expressions such as $A^2(X^2 - BX + AD)$ have meaning in terms of matrix operations. This algebra shares many of the fundamental properties of ordinary algebra; some of these are the associative laws for addition and multiplication $(A + B) + C = A + (B + C)$ and $A \cdot (B \cdot C) = (A \cdot B) \cdot C$, and the distributive law, $A \cdot (B + C) = A \cdot B + A \cdot C$. One of the striking differences is the failure of the commutative law for multiplication, that is, $A \cdot B \neq B \cdot A$ for matrices A and B in general (the commutative law of addition does hold). Such a restriction has a drastic effect on algebraic technique; for example, while xax can be replaced by $ax \cdot x$, or ax^2, in ordinary algebra, the expression XAX cannot be so modified for matrices. Another striking difference is the failure of the principle of ordinary algebra which asserts that a product can equal 0 only if at least one factor is 0; the product of two nonzero matrices can equal the zero matrix (the matrix whose entries are all zeros). Matrices make up a type of "number" system known as a **ring**.

The theory of matrices was developed in the middle of the nineteenth century by Cayley, and is a central feature of higher algebra. One of its principal uses is in geometry, where a matrix can denote a *transformation* of a space. Matrices are important in the study of vectors, and in many parts of pure and applied mathematics.

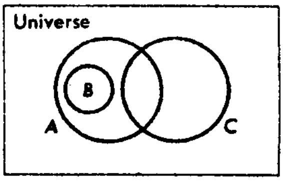
Ref. [51, 55].

Algebra of Propositions (Syn.: Propositional calculus; Sentential calculus) The part of the logic which treats propositional forms built out of the sentence connectives "and," "or," "if . . . , then . . . ," and "not"; particularly, the logical validity of such forms. For example, the proposition "The earth is round or the earth is not round" is recognized as being true (logically valid), not because of any property of the earth but because of the form of the sentence; this form is symbolized "P or (not P)," and any proposition with this sentence structure is true. The connectives define the *conjunction*, "P and Q," the *disjunction*, "P or Q," the *conditional*, "If P, then Q," and the *denial* "not P." Out of these, more complicated forms can be put together; examples are "(P and Q) or (not Q)," "If (P or R), then Q." The law of excluded middle asserts the (logical) validity of "P or (not P)" (illustrated in the example above); the law of contradiction asserts the validity of "not [P and (not P)]" (that is, that a proposition and its denial are not both true). Propositions are *logically equivalent* in case they say the same thing in different propositional forms. An example is given by the law of double negation, which asserts that "not (not P)" is (logically) equivalent to "P." The law of contraposition asserts that the form "If P, then Q" and the form "If (not Q), then (not P)" are equivalent. For example, the sentence "If the lot is small, then the house is small" is equivalent to "If the house is not small, then the lot is not small."

In the algebra of propositions, symbols are introduced for sentence connectives: "$P \wedge Q$" is often used for "P and Q," "$P \vee Q$" for "P or Q," "$P \rightarrow Q$" for "if P, then Q," and "$\sim P$" for "not P." In this notation, the sentence "It is not the case that all is lost and the gold is not safe" takes the form "$\sim(P \wedge (\sim Q))$." With variables "P," "Q," etc., standing for propositions, and symbols "$\wedge$," "$\vee$," etc., for connectives, the study of propositional forms assumes a highly algebraic aspect; the algebra of propositions, which evolved in the second half of the nineteenth century, is a type of mathematical system called a **Boolean algebra**.

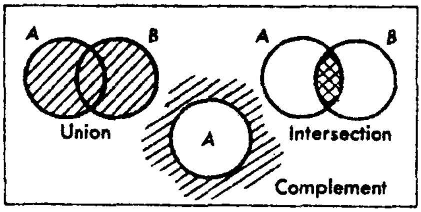
Ref. [18, 39, 53, 59, 68].

Algebra of Sets (Syn.: Algebra of classes; Calculus of classes) A part of logic (or mathematics) that treats classes, or **sets**, of things, and operations and relations among these sets; here, symbols (variables) such as "A," "B," etc., are used to stand for sets, and expressions are formed out of these and certain signs for operations and relations (much as expressions are formed with variables denoting numbers in ordinary algebra). The *universe*, or universal set, is understood as the set of all individual things, and the *null*, or empty set, as the set with no individuals; these are sometimes denoted by 1 and 0, respectively. For example, in a particular use, all integers might be taken as the universe, and a set would mean any collection of integers; the null set might arise as the set of integers whose squares are negative. In plane geometry, all the points in the plane might be regarded as the universe, and a set would be any collection of points; the null set can arise as the set of points of intersection of a circle and a line that does not meet it. Various *relations* among sets arise. That A and B are *equal*, written "$A = B$," means that A and B are the same sets (have exactly the same members); for example, the set of equilateral triangles equals the set of equiangular triangles. That set A *is included in* set B, or A is a subset of B, written "$A \subset B$," means that every member of A is a member of B; for example, the set of powers of 3 is a subset of the set of odd numbers. That two sets *overlap* means that they have at least one member in common; that they are *disjoint*



means that they have no members in common. These relations can be represented geometrically, as in the figure; here the universe is represented by the interior of the rectangle and the sets A, B, C by the interiors of the circles. In the figure, B is included in A, A and C overlap, and B and C are disjoint; such geometric representations are called *Venn diagrams.* Various general statements hold for these relations; among these are the following. The null set is a subset of every set, while every set is a subset of the universe. The relation of inclusion between sets is a transitive relation; that is if $A \subset B$ and $B \subset C$, then $A \subset C$. Equality and inclusion are connected by the fact that $A = B$ just in case $A \subset B$ and $B \subset A$.

Operations on sets include union and intersection (see **Property**). The *union*, or *sum*, of A and B is the smallest set containing both; it is denoted by "$A \cup B$." Its members are the members of A taken together with the members of B; for example, the union of {1, 3, 5} and {2, 3, 5, 7} is {1, 2, 3, 5, 7,}. The *intersection*, or *product*, of A and B is the largest set common to both; it is denoted by "$A \cap B$". Its members are those common to both sets (the intersection of the given sets in the last example is {3, 5}). Finally, the *complement* of A, denoted by A', is the set consisting of all elements of the universe which are not in A; for example, the com-



plement of the set of even whole numbers in the universe of all whole numbers is the set of odd whole numbers. Many general statements hold involving operations on sets. The union of any set A with the null set is A; the union of A with the universe is the universe. The intersection of A with the null set is the null set, and with the universe is A itself. The complement of the universe (null set) is the null set (universe). Union and intersection are commutative and associative; in this respect they are analogous to addition and multiplication in ordinary algebra. However, other properties are strikingly different; for example, the equations $A \cup A = A$ and $A \cap A = A$ hold for all sets A; their analogs in arithmetic, $x + x = x$, and $x \cdot x = x$, do not hold for all numbers x. *De Morgan's laws* are the following equalities, mixing the three operations $(A \cap B)' = A' \cup B'$ (the complement of an intersection is the union of the complements), and $(A \cup B)' = A' \cap B'$. Operations on sets can be connected with relations between sets. Two sets are disjoint in case their intersection is the null set. If one set is a subset of the other, then the complement of the second is a subset of the complement of the first; that is, if $A \subset B$, then $B' \subset A'$.

The algebra of sets was originated by Boole in the middle of the nineteenth century; it is a type of mathematical system known as a **Boolean algebra**.

Ref. [18, 39, 53, 68].

Algebra of Vectors The symbolic study of certain operations and relations among vectors. See **Vector, Vector space.**

Algebraic Referring to concepts or methods of algebra. Sometimes used in connection with signed numbers, in contrast to unsigned numbers, as in "algebraic addition," "algebraic angle," or "algebraic multiplication."

Algebraic Addition In arithmetic, the addition of signed numbers, that is, of positive and negative numbers. Such addition can be reduced to the addition (or subtraction) of positive numbers by the following law of signs:

(1) $(-a) + (-b) = -(a + b)$;
(2) $a + (-b) = a - b$;
(3) $a + (-b) = -(b - a)$.

Rule **(1)** implies that the sum of two negative numbers is the sum of the numerical values with the minus sign prefixed; for example,

$$(-3) + (-8) = -(3 + 8), \text{ or } -11.$$

Rules **(2)** and **(3)** imply that the sum of a positive and a negative number is obtained by subtracting the smaller numerical value from the larger numerical value, and prefixing the sign of the number with the larger numerical value; e.g., $5 + (-2) = 5 - 2$, or 3, and $2 + (-5) = -(5 - 2)$, or -3. A law of signs is also available for **subtraction**.

Algebraic Equation An equation with each side being an algebraic expression. For example, $3 - \sqrt{x} = 5x^2 + y$; is an algebraic equation; the equation $3 - \sin x = 5x^2 + y$, is not algebraic because the term $\sin x$ is not (it is a transcendental equation).

Algebraic Expression Any expression in variables and constants which designates numbers and involves only the application of algebraic operations; these are the expressions encountered in ordinary algebra. Special types of algebraic expressions are given particular names. A *monomial* involves only multiplication between variables and constants, such as $3xy$ and $-5ax^2$; monomials, which differ only in their numerical factors, such as $3ax$ and $7ax$, are *similar terms,* or *like terms* (a monomial is sometimes called a "term"). A *binomial* is a sum of two monomials (terms), as in $2x + ay^2$; a *trinomial* is a sum of three monomials. A *multinomial* or *polynomial* (or a *rational integral expression*) is a sum of any number of monomials; for example, $ax^2 - \frac{1}{2}xy + 3y - 5y^2z$ is a rational integral expression whose terms are ax^2, $-\frac{1}{2}xy$, $3y$, and $-5y^2z$. A *rational expression* is a quotient of multinomials, or an algebraic expression which can be transformed to such a quotient. This type of expression may involve any algebraic operation on the variables but root extraction; for example, $2 + [x/(x+1)]$ is a rational expression [it can be transformed to $(3x + 2)/(x + 1)$].

An *irrational expression* is one which is not rational; it involves root extraction of an expression containing a variable, as in $\sqrt{2x + 1} - 5y$.

The variables of an algebraic expression are sometimes singled out, as in the terminology "a rational expression in x," or "an algebraic expression in x and y." For example, $x\sqrt{y} - 2x^2$ is a polynomial in x but not in y; it is an algebraic expression in either variable or both. See **Transcendental.**

Algebraic Function An *explicit algebraic function* (or, simply, *algebraic function*) is a function whose value is given by an algebraic expression; for example, $f(x) = 3x^2 + 2\sqrt{x}$ specifies such a function. These algebraic functions may be classified according to the algebraic expressions which define them; for example, a function is *rational integral, rational,* or *irrational* according to whether the expression is of the same type.

In advanced mathematics, an algebraic function is taken to mean a correspondence from values of x to values of y as determined by a polynomial equation $P(x, y) = 0$; an example is

$$2xy^3 + x^2y - 3xy + 5x^3 + 7 = 0.$$

This is sometimes called an *implicit function,* to distinguish it from the first type. The theory of algebraic functions is an extensive one in advanced mathematics.

Algebraic Identity An **identity** which is an algebraic equation.

Algebraic Multiplication In arithmetic, the multiplication of signed numbers, that is, of positive and negative numbers. Such multiplication can be reduced to multiplication of positive numbers by use of the following *law of signs*:

(1) $(-a)\cdot(-b) = a\cdot b$;
(2) $(-a)\cdot b = -(a\cdot b)$.

These provide the rule that like signs give "plus," and unlike signs give "minus"; e.g., $5\cdot(-3) = -(5\cdot 3)$, or -15, and $(-4)\cdot(-\frac{3}{5}) = 4\cdot(\frac{3}{5})$, or $\frac{12}{5}$. A similar law of signs is also available for **division**.

Algebraic Number A real number (or complex number) which is a solution of some polynomial equation whose coefficients are rational numbers; it is an *algebraic integer* if the coefficients are ordinary integers and the leading coefficient is one. The theory of algebraic numbers is an important part of higher algebra. See **Integer, Number field**. Ref. [16, 21].

Algebraic Operations In ordinary algebra, the operations of addition, subtraction, multiplication, division, root extraction, and raising to an integral or fractional power. The nonalgebraic, or *transcendental,* operation is illustrated by the logarithm; the definition of the logarithm rests upon the limit process of the calculus.

Algorithm Usually, an explicit method of computation which proceeds in a step-by-step manner, repeating an underlying process. One example is the method of square root calculation taught in school; another is the familiar procedure of long division. Sometimes "algorithm" is used for any rule of calculation or explicit method of solution. See **Euclidean algorithm**.

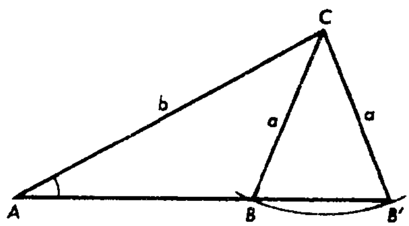
Alternate Angles of Transversal See **Transversal**.

Alternating Series An infinite series whose successive terms are alternately positive and negative. If the terms are decreasing in numerical value and have the limit 0, then the series is necessarily convergent; an example is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, with n^{th} term $(-1)^{n-1}/n$.

Alternation In *logic,* the same as **disjunction**; in *proportions,* the deduction of $a/c = b/d$ from $a/b = c/d$.

Altitude In geometry, generally, a line segment (or its length) which measures the height of a figure. See particular figures such as **Cone, Triangle,** etc.

Ambiguous Case In *plane trigonometry,* the case of the **solution of the triangle in the plane** where the given data lead to two solutions. It occurs when two sides and the angle opposite one of them are given, as sides a, b, and angle A, in the figure;



each of the triangles ABC, $AB'C$ satisfies the given conditions. In *spherical trigonometry,* the ambigious case in the **solution of the oblique spherical triangle** occurs when two sides and an angle opposite one of them are given, or when two angles and the side opposite one of them are given (this latter case is peculiar to the spherical triangle).

Amplitude of Complex Number (Syn.: Argument; Phase) The angle of rotation about the origin of the positive x-axis into the point with rectangular coordinates (a, b), representing the complex number $a + bi$.

Amplitude of Periodic Function See **Periodic function**.