WEIGHT FUNCTIONS AND STRESS INTENSITY FACTOR SOLUTIONS

Xue-Ren Wu
and
A. Janne Carlsson

Pergamon Press

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XUE-REN WU
*Institute of Aeronautical Materials,
Beijing, China*

and

A. JANNE CARLSSON
*Royal Institute of Technology
Stockholm, Sweden*

PERGAMON PRESS
OXFORD · NEW YORK · SEOUL · TOKYO

U.K. Pergamon Press plc, Headington Hill Hall, Oxford OX3 0BW, England

U.S.A. Pergamon Press, Inc, 395 Saw Mill River Road, Elmsford, New York 10523, U.S.A.

KOREA Pergamon Press Korea, KPO Box 315, Seoul 110-603, Korea

JAPAN Pergamon Press, 8th Floor, Matsuoka Central Building, 1-7-1 Nishishinjuku, Shinjuku-ku, Tokyo 160, Japan

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First edition 1991

Library of Congress Cataloging in Publication Data
Wu, Xue-Ren.
Weight functions and stress intensity factor solutions/
Xue-Ren Wu and A. Janne Carlsson.—1st ed.
p. cm.
Includes bibliographical references.
1. Fracture mechanics. 2. Elastic solids. 3. Strains and stresses. I. Carlsson, Janne, 1932- . II. Title.
TA409.W8 1991 620.1'126–dc20 91-26652

British Library Cataloguing in Publication Data
Wu, Xue-Ren
Weight functions and stress intensity factor solutions.
I. Title II. Carlsson, A. Janne,
620.1126

ISBN 0-08-041702-7

Printed in Great Britain by BPCC Wheatons Ltd, Exeter

Preface

Fracture mechanics has become an indispensable tool in a number of important technical areas for the design and safe operation of damage-tolerant structures, and the development and use of advanced materials. One of the essential ingredients in fracture mechanics based analysis is the stress intensity factor, the characterizing parameter for the crack tip field in a linear elastic material. The determination of stress intensity factors for cracked members has therefore been the subjects of intensive research and work in fracture mechanics during the past three decades.

The weight function method is a very powerful and cost-effective method for the evaluation of stress intensity factors and crack opening displacements, thus greatly facilitating the application of fracture mechanics. Compared with most current analytical and numerical methods, it has distinct advantages: it features versatility, efficiency, reliability and is easy to use. The method is especially attractive when a large number of stress intensity factors are desired for multiple load conditions. This book reflects the authors' research in this particular area in recent years. It attempts to systematically present the weight function method for two-dimensional crack problems in a unified manner using closed form analysis. A very large number of stress intensity factors are given, all of which have been generated by the authors using the analytical weight functions in the book. Every effort has been made to ensure that the solutions are reliable.

The book can be divided into three parts: Part I (Chapter 1) gives the theoretical background and overview of the weight function method. Part II (Chapter 2 through 16) constitutes the bulk portion of the book, giving details of the weight functions for various geometries and a large number of stress intensity factor solutions in graphical and/or tabular form; center crack(s) are treated in Chapters 2 to 4, edge cracks in Chapters 5 to 16. Part III (Chapter 17) deals with the determination of crack opening displacements, Dugdale model solutions and crack opening areas.

The book is primarily intended to serve as a useful reference for researchers and engineers concerned with fracture and fatigue of materials and structures. The reader is expected to have basic knowledge of fracture mechanics.

The authors wish to express their gratitude to many of their colleagues in various organizations, who have helped and assisted in the production of the manuscript by providing excellent working conditions, making good suggestions on the presentation style, spending their valuable time on reading and commenting for the improvement of the book. The final assembling of the material and the camera-ready typescript were completed during one of the authors' (X R Wu) academic visit to the Royal Institute of Technology, Stockholm; he gratefully acknowledges the support of both his host and home organization. Finally, we thank our families for the continued encouragement and support throughout the work.

X. R. Wu and A. J. Carlsson
Stockholm, April 1991

User Guide

The aim of this book is twofold: the first being to provide a theoretical background of the weight function method for analysis of two dimensional crack problems based on analytical approaches; the second being to present a large number of stress intensity factors (K_I) as well as closed form weight functions for various crack geometries of practical significance.

Throughout the book, the analysis has been carried out in terms of normalized quantities; the normalization is necessary in order to avoid lengthy expressions. For this purpose some of the notations do not follow the customary ones: all the real length dimensions are denoted by capitals, e.g. the crack length by A, the coordinate by X and the crack opening displacement by U. For each crack geometry a characteristic length dimension, W, is chosen for normalization. The most frequently used normalized quantities are

$$a = A/W \qquad x = X/W \qquad u = U/W \tag{1}$$

In most cases the stress intensity factors are written in the form

$$K = f\sigma\sqrt{\pi a W} \tag{2}$$

where f is the non-dimensional stress intensity factor and σ is a normalizing stress which can be chosen freely as long as consistency is kept throughout the analysis. This σ is not to be confused with the crackline stress distribution denoted by $\sigma(x)$.

Because f is a function of the normalized crack length a, it is more convenient to work with the normalized geometry in which W is set to be unity, as shown in the first figure in all the chapters from 2 to 16. In most of the insets of the f-plots this W has been put equal to 1. The real dimension W, however, must be used when calculating K according to e.g. eq (2).

For users' convenience, the same structural form has been adopted for the chapters concerned with stress intensity factors (Chapter 2 to 16). These chapters are independent of one another and each chapter consists of three sections: *1.* Determination of weight function(s), *2.* Stress intensity factors for basic load cases, *3.* Application examples containing K-solutions to various load cases.

The use of the book, however, is not restricted to the given examples. The possibility of generating f-solutions for new load cases is unlimited. In most cases, solutions can be easily obtained without resorting to numerical integration. In this regard, the reader will find the basic solutions very useful.

The weight function method is to be used in conjunction with the superposition principle in linear elastic fracture mechanics. When using the weight functions and the basic solutions in this book, the first step is to determine $\sigma(X)$ (frequently referred to as crackline stress or crack face pressure/loading) along the prospective crack line in the body *without the crack.* Because of the absence of the crack, $\sigma(X)$ can be easily obtained by conventional analytical or numerical methods. This $\sigma(X)$ must then be transformed to $\sigma(x)$ using the normalized coordinate x. Depending on the nature of the crackline stress the corresponding f can be evaluated using various options.

- For most cases of continuously distributed crack face loadings, f can be readily determined by superposition of the f_n-solutions in the relevant chapter. To this end, the crackline stress $\sigma(x)$ is first expressed, by least square fit, in polynomial form:

$$\sigma(x) = \sum_{n=0}^{N} S_n x^n \ , \qquad x \geq 0 \tag{3}$$

(It is important to note that to use the f_n-solutions, the stress over the *entire* crack length a in consideration must be represented by a *single* polynomial.)

The non-dimensional stress intensity factor f is then determined by the simple arithmetic:

$$f = \sum_{n=0}^{N} S_n f_n \tag{4}$$

- If $\sigma(x)$ is not amenable to polynomial representation because of its discontinuities or strong variations, then the piecewise linearization method can be used. The $\sigma(x)$-distribution $(0 \leq x \leq a)$ is discretized into linear segments, and the resultant f is obtained by summation of the contribution from each linear segment.

The weight function method itself is exact. The main sources of error in the determination of stress intensity factors are: the accuracy in the reference solutions and the number of terms contained in the weight function. For each crack geometry, the accuracy of the weight functions presented in this book has been examined carefully in the given range of the non-dimensional crack length a. An important advantage of the weight function method is that stress intensity factors are obtained through integration, an operation usually reducing the possible errors.

Contents

1

Theoretical Background and Overview

1.1 Introduction

Fracture mechanics, being one of the most active research areas in the field of solid mechanics since the middle of the 1950's, has developed into an important branch of applied mechanics. The rapid development of fracture research has mainly been driven by the technological need for ensuring structural safety and for developing new advanced materials. Today the use of fracture mechanics and damage-tolerance analysis is almost imperative in areas where structural failure could endanger human lives or lead to major economic losses. Among others, aircraft/aerospace, nuclear energy/power generation, off-shore and transport are the notable industry sectors where fracture mechanics is extensively used as the theoretical basis for the development of fracture control plans, design codes and inspection regulations. These activities ensure that operational safety and structural integrity are adequately maintained during the expected life span.

Fracture mechanics focuses its attention on assessing, in a quantitative manner, the behavior of cracks or cracklike defects in materials and structural components, which may be introduced during the material processing and fabrication or in service by damage due to e.g. overloading, fatigue or environmental effects. At present there are still a multiple number of very important issues yet to be solved in this field, such as elastic-plastic, dynamic and high temperature problems. However, linear elastic fracture mechanics (LEFM) is now well established, and it is the part of fracture mechanics which has been most widely used in the industry.

1.1.1 *Linear elastic fracture mechanics and the stress intensity factor*

Compared to many other branches of solid mechanics, fracture mechanics has a relatively short history. Although early attempts were made in the 1920's to understand brittle fracture using energy methods, intensive studies and subsequent widespread applications of fracture theory had not taken place until the mid-1950's when the concept of stress intensity factor K was first introduced, and the relation between the crack tip field parameter K and the energy release rate G established by the classical work of Irwin [1.1] and Williams [1.2]. A survey paper by Paris and Sih [1.3] has summarized the principal results from earlier studies of linear elastic analysis of the crack tip fields.

Crack tip fields can be divided into three basic types, each associated with a local mode of deformation, Fig. 1.1. The mode of deformation is characterized by the direction of the potential motion of the crack surfaces in relation to the original crack plane, Fig. 1.1; they are:

Mode I: in-plane opening causing normal separation of the crack faces;

Mode II: in-plane shearing leading to relative in-plane sliding of the crack faces perpendicular to the crack front;

Mode III: antiplane shearing leading to relative out-of-plane sliding of the crack faces parallel to the crack front.

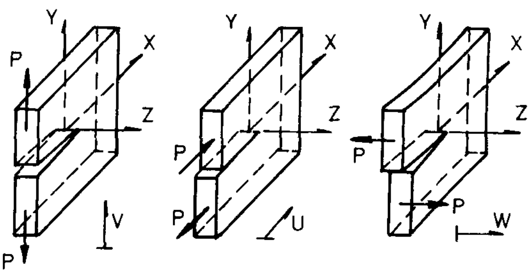

Fig.1.1. Three basic modes of crack surface displacements.

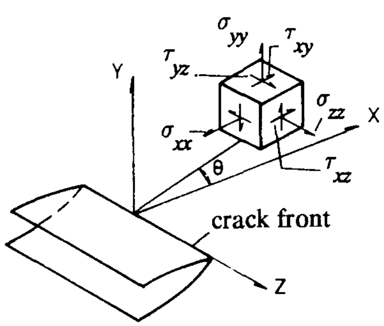

Fig. 1.2. Coordinate system and crack tip stress components.

The crack tip ($r \to 0$) stress and displacement fields, in a rectangular coordinate system are, Fig. 1.2

Mode I:

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \left[1 - \sin\frac{\theta}{2} \sin\frac{3\theta}{2} \right] + \sigma_{xx0} + O(\sqrt{r})$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \left[1 + \sin\frac{\theta}{2} \sin\frac{3\theta}{2} \right] + O(\sqrt{r})$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \sin\frac{\theta}{2} \cos\frac{3\theta}{2} + O(\sqrt{r})$$

$$\tau_{xz} = \tau_{yz} = 0; \quad \sigma_{zz} = 0 \text{ (plane stress)}, \quad \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) \text{ (plane strain)}$$

$$u = [K_I (\tfrac{r}{2\pi})^{\frac{1}{2}} / (4\mu)] \left[(2\kappa - 1) \cos\frac{\theta}{2} - \cos\frac{3\theta}{2} \right] + O(r)$$

$$v = [K_I (\tfrac{r}{2\pi})^{\frac{1}{2}} / (4\mu)] \left[(2\kappa + 1) \sin\frac{\theta}{2} - \sin\frac{3\theta}{2} \right] + O(r)$$

$$w = -\frac{\nu}{E} \int (\sigma_{xx} + \sigma_{yy})\, dz \text{ (plane stress)}, \quad w = 0 \text{ (plane strain)} \tag{1.1}$$

Mode II:

$$\sigma_{xx} = -\frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2} \left[2 + \cos\frac{\theta}{2} \cos\frac{3\theta}{2} \right] + O(\sqrt{r})$$

$$\sigma_{yy} = \frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2} \cos\frac{\theta}{2} \cos\frac{3\theta}{2} + O(\sqrt{r})$$

$$\tau_{xy} = \frac{K_{II}}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \left[1 - \sin\frac{\theta}{2} \sin\frac{3\theta}{2} \right] + O(\sqrt{r})$$

$$\tau_{xz} = \tau_{yz} = 0; \quad \sigma_{zz} = 0 \text{ (plane stress)}, \quad \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) \text{ (plane strain)}$$

$$u = [K_{II} (\tfrac{r}{2\pi})^{\frac{1}{2}} / (4\mu)] \left[(2\kappa + 3) \sin\frac{\theta}{2} + \sin\frac{3\theta}{2} \right] + O(r)$$

$$v = -[K_{II} (\tfrac{r}{2\pi})^{\frac{1}{2}} / (4\mu)] \left[(2\kappa - 3) \cos\frac{\theta}{2} + \cos\frac{3\theta}{2} \right] + O(r)$$

$$w = -\frac{\nu}{E} \int (\sigma_{xx} + \sigma_{yy})\, dz \text{ (plane stress)}, \quad w = 0 \text{ (plane strain)} \tag{1.2}$$

Mode III:

$$\tau_{xz} = -\frac{K_{III}}{\sqrt{2\pi r}} \sin\frac{\theta}{2} + O(\sqrt{r})$$

$$\tau_{yz} = \frac{K_{III}}{\sqrt{2\pi r}} \cos\frac{\theta}{2} + O(\sqrt{r})$$

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \tau_{xy} = 0$$

$$u = v = 0, \; w = [K_{III}(\frac{2r}{\pi})^{\frac{1}{2}}/\mu] \sin\frac{\theta}{2} + O(r) \tag{1.3}$$

where the subscripts *I*, *II*, and *III* denote the mode of loading, μ is the shear modulus, $\kappa = (3 - \nu)/(1 + \nu)$ for plane stress and $\kappa = 3 - 4\nu$ for plane strain, with ν being Poisson's ratio.

Two features are evident from the field equations (1.1-1.3):

i: For any given mode of deformation, the angular dependence of the crack tip stresses and that of displacements are universal, i. e. the θ-distributions of the crack tip field quantities are functions of the deformation mode only.

ii: The crack tip stresses all contain a $1/\sqrt{r}$ singularity whose intensity is determined by a single parameter, the stress intensity factor K. All other aspects of loading and geometry including the crack size can only affect the behavior of the cracked body through the stress intensity factors.

The singular stress is the first and the dominating term in a power series expansion of the stress field around the crack tip [1.2]. We note that for mode I loading, there is a second constant term σ_{xx0} in σ_{xx}. This non-zero component is parallel to the crack plane and is often referred to as the T-stress. The T-stress is geometry dependent and has been given by Larsson and Carlsson for some specimen geometries [1.4].

The singular stress distribution at the crack tip is obviously a result of the mathematical idealization that the crack is infinitely sharp and the material is linearly elastic everywhere. In reality, the stresses in the vicinity of the crack tip will be relieved by plastic flow, or other non-linear deformation. However, under the condition of *small scale yielding* [1.5], meaning that yielding is confined in a region small compared to the crack length and other body dimensions, the corresponding stress distributions surrounding this small inelastic zone are still adequately described by the dominant singular term in the elastic solution and its coefficient, the stress intensity factor K. Therefore the stress intensity factor provides a one-parameter characterization of the crack tip field in a linear elastic body, provided certain geometrical requirements are met [1.6]. (Note that the T-stress mentioned above has an influence on the plastic deformation at the crack tip, thereby limiting the validity of LEFM as has been pointed out by Larsson and Carlsson [1.4], and Rice [1.7]. It was also shown that the T-stress effect on other crack tip parameters, such as the crack opening displacement and