NUCLEAR

PHOTO-DISINTEGRATION

BY

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PREFACE

My research work on nuclear photo-disintegration started in 1949 at Cornell University with H. A. Bethe, and also P. Morrison. In the summer of 1956, I gave a series of lectures on "Electromagnetic Interactions with Nuclei" at the University of Mexico. The lecture notes prepared by M. L. Rustgi and L. Estrada formed the nucleus of the manuscript for this book; while vigorous discussions with physicists there greatly clarified my point of view concerning nuclear models.

The bulk of the writing of the manuscript was done while I was a Guggenheim Fellow working at the University of Birmingham. My main emphasis in this work on the nuclear photoeffect, and in the manuscript, now became a critical analysis of various nuclear models used for these calculations, and consideration of the relations among various nuclear models. The penetrating discussions with Prof. Peierls, and many others of the Department of Mathematical Physics, proved highly illuminating.

I also wish to thank many physicists for telling me of their theoretical or experimental results prior to publication, and for detailed discussions concerning their work. Prof. D. H. Wilkinson provided very many constructive criticisms after reading my Birmingham draft of the manuscript. I am also grateful to G. E. Brown, J. Dabrowski, A. M. Lane, and G. G. Shute for their criticisms of this draft. Finally, I wish to thank M. Razavy, O. Rojo, and my wife for their help in the preparation of figures, and in proof reading.

<div align="right">J. S. L.</div>

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CONTENTS

INTERACTIONS BETWEEN CHARGED PARTICLES AND RADIATION

1.1. Introduction

IN our discussion of the theory of nuclear photo-disintegration we will be guided, as far as possible, by analogies to the atomic photo-effect. We therefore spend the next five sections in a survey of methods for calculating the atomic photo-effect. The last three sections give an introduction to theoretical and experimental work on the nuclear photo-effect. As there is a high density of equations in this survey, we shall in this section attempt to give a physical interpretation of the major results derived below.

In cases where the analogy to the atomic photo-effect fails, we have recourse to a phenomenological approach to nuclear physics. That is, we try to interpret nuclear phenomena at moderate energies (say less than 100 MeV excitation) in terms of non-relativistic wave functions involving only nucleon coordinates. We attempt to avoid discussion of mesonic effects although they are certainly of importance at high energies.

We have not attempted a complete survey either of theoretical or of experimental work. Our use of experimental data in tables or figures is for illustrative purposes only. (See Toms [197] for a bibliography of the literature; Bishop and Wilson [27] for experimental data; and Sachs [176] and Blatt and Weisskopf [28] for theoretical material.)

There are three reasonable approaches to the calculation of transition probabilities for the atomic photo-effect: the original Heisenberg approach of postulating the transition probability in terms of matrix elements; the perturbation-theoretic approach in which the electromagnetic field is treated classically; and the approach of quantum electrodynamics [98]. We have chosen the second approach as not too advanced, and yet rigorous enough for our purposes.

Consider the case of absorption of electromagnetic energy in which an atom is excited from the ground to a higher discrete state. In section 1.2 we calculate the transition probability integrated over the absorption line. Alternatively this integrated transition probability can be expressed as an absorption cross section integrated over the line, or as an oscillator strength. If the atom is originally in the higher discrete state, we are concerned with the transition rate for spontaneous photon emission, which is proportional to the oscillator strength. In the photoeffect the atom is excited from a discrete to a continuum state. In this case we consider either the oscillator density or the related absorption cross section.

In section 1.3 we justify our use of the oscillator strength (or density) by analogies with three different classical calculations of the interaction of electromagnetic waves with a charged oscillator: the integrated power absorption, Thomson scattering, and the polarizability.

We then apply "sum-rules" to calculate various statistical moments of the oscillator strength distribution and corresponding mean energies. These sum-rules all use the closure property of matrix mechanics: $\sum_n A_{0n} B_{n0} = (AB)_{00}$, where A and B are any two matrices and we sum over all excited (discrete or continuum) states n. These sum-rules prove useful, because the answer is expressed using only expectation values for the ground (0) state. Thus sum-rule calculations can clarify to what extent a result derived from some particular model is peculiar to that model, or on the other hand may be model-independent. For example, two different models that use the same wave function for the ground state but different wave functions for excited states will give identical sum-rule results.

It may seem surprising that the properties of the ground state alone are sufficient for sum-rule calculations which determine the major properties of the photo-effect cross section [130]. We can understand this result from the Schrödinger equation. The ground state wave function determines the Hamiltonian for which it is the lowest eigenfunction. This Hamiltonian in turn determines the wave functions of all excited states, and

hence all matrix elements needed for a complete calculation of the oscillator strengths for all energies. While this argument is rigorous in the atomic case (non-relativistic approximation) it must be modified in the nuclear case, since the nuclear Hamiltonian is différent for different states of the nuclear system. (For instance, the Majorana exchange force has a different sign in the ground state of the deuteron than in the excited state reached by photon absorption.) It turns out that we can modify our sum-rules to account for the state-dependence of the Hamiltonian, and again express our answer in terms of the ground state expectation value of a more complex expression than that used in the atomic case.

In section 1.5 we give an introduction to the multipole expansion of the electromagnetic field, and the selection rules for the various multipoles. In the non-relativistic limit (photon energy much smaller than the mass energy mc^2 of the particles considered) the electric field is almost constant over the atom. This electric dipole ($E1$) approximation is made in most of our calculations. The $E1$ approximation selects out from the photon field the part that contains one unit of angular momentum and odd parity: hence the selection rule that the atomic system must change its angular momentum by one unit (in the sense of the vector model for addition of angular momenta) and must change its parity. Other multipoles select out other angular momenta and parity. For example, the magnetic dipole term has one unit of angular momentum and even parity, while the electric quadrupole term has two units of angular momentum and even parity. Magnetic dipole and electric quadrupole have similar orders of magnitude for their transition rates, and are each forbidden transitions with transition rates small compared to the allowed $E1$ transitions.

The nuclear photo-effect differs from the atomic photo-effect since protons and neutrons have essentially equal masses. We are concerned with the nuclear dipole moment caused by proton displacements. We separate this dipole moment into one part due to the motion of the centre of mass of the entire nucleus, and a second part due to displacements relative to the

centre of mass. We rewrite the second part, which is responsible for photon absorption, in terms of both proton and neutron coordinates, thus introducing the "effective charges" of $(N/A)e$ for protons and $-(Z/A)e$ for neutrons.

The nuclear photo-effect presents a fundamental problem, due to the exchange forces among the nucleons. These exchange forces create an ambiguity as to the relevant operator for interaction of the protons with the electromagnetic field. This ambiguity is resolved by Siegert's theorem, which states that (for $E1$ transitions) we should consider the interaction of the dipole moment caused by point-protons with the electric field. Meson currents exist in a nucleus; but it should be possible to specify the charge density in a nucleus without explicit reference to mesons. We express the $E1$ interaction in terms of the charge density, thus giving a phenomenological derivation of Siegert's theorem.

The theoretical integrated cross section for nuclear photo-disintegration is increased above the "classical value" of 0·060 (NZ/A) MeV-barns. This increase was first calculated introducing an attractive neutron-proton exchange force in the nuclear Hamiltonian. It has been re-interpreted in terms of the velocity-dependence of the nuclear shell-model potential. A third interpretation of the increase in the integrated cross section is given using dispersion theory, in which the cross section for the nuclear photo-effect, integrated up to the threshold for meson production, is expressed in terms of the difference of relevant integrated photo-meson cross sections. The relations among these three approaches are not yet known. To relate the first and second approaches we need a theory deriving the nuclear shell-model potential from the two-body nuclear force. To relate the first and third, we need a theory deriving the two-body nucleon force from a meson theory.

In the last section of this chapter we give a summary of experimental techniques, difficulties, and results in measurements of nuclear photo-disintegration.

In Chapter II we discuss experimental data and theoretical

calculations on the deuteron photo-effect. The good agreement found at present between experiment and theory gives us some confidence that we know both the wave functions (for a nucleon force with repulsive core and spin-orbit potential) and the operator for the nucleon's interaction with the electromagnetic field. Above about 150 MeV photon energy, explicit mesonic effects are of importance; but mesonic effects seem to be of minor importance below about 100 MeV.

In the next chapter we use sum-rules for calculations of various statistical moments and mean energies for the nuclear photo-effect. We find that the general features of the nuclear photo-effect are insensitive to the model assumed for the nuclear ground state, and are in fair agreement with experiment.

In Chapter IV we discuss briefly discrete transitions in the nuclear photo-effect, paying special attention to $E1$ transitions. We find rough agreement between shell-model estimates and experimental data on transition probabilities.

In Chapter V we consider the three principal models proposed for the nuclear photo-effect: the shell model, the collective model, and for high energies the quasi-deuteron model. We conclude that neither the shell or collective model is literally correct. Either model can be used to calculate the absorption cross section as a function of photon energy; but the main characteristics of the nuclear photo-effect are already given by model-insensitive sum-rules. However, the models have been of tremendous psychological help, in that they give a physical feeling for the problem. This physical feeling is absent in the sum-rule approach. Further work is needed to combine the high energy quasi-deuteron model with either the shell or collective model valid for moderate energies.

In the final chapter we discuss calculations and experiments on the products of nuclear photo-disintegration: the branching ratios for emission of various particles, their energy spectra, and their angular distributions. We find that the probable reactions are in general accord with the statistical model for decay of the compound nucleus. Deviations from the statistical model can be explained in terms of resonance direct emission.

At high energies neutron-proton coincidences are in good agreement with the predictions of the quasi-deuteron model.

1.2. The oscillator strength

If a quantum-mechanical system, originally in its ground state, 0, is perturbed for a time t, then the probability $|C_n|^2$ of finding the system in a discrete state n is given by time-dependent perturbation theory [151] as

$$|C_n(t)|^2 = \left| -\frac{i}{\hbar} \int_0^t e^{i\omega_{0n}t'} [H'(r, t')]_{0n} \, dt' \right|^2. \tag{1.1}$$

Here H' is the perturbation in the Hamiltonian, the subscripts 0 and n denote the matrix element between stationary states, and the angular frequency ω_{0n} is given by the Planck relation

$$\omega_{0n} = (E_n - E_0)/\hbar. \tag{1.2}$$

In the electric dipole ($E1$) approximation, the perturbation H' is due to a spatially constant electric field along the z axis of amplitude $\mathscr{E}$, and angular frequency ω. We shall consider the perturbation due to the electric field interacting with a single particle of charge e:

$$[H'(r, t')]_{0n} = -e[z]_{0n} \cos \omega t'. \tag{1.3}$$

Substituting in equation 1.1 and carrying out the integration over dt', we find

$$|C_n(t)|^2 = e^2\mathscr{E}^2(z_{0n})^2 \left[\frac{\sin^2 (\omega - \omega_{0n})t/2}{\hbar^2(\omega - \omega_{0n})^2} + \frac{\sin^2 (\omega + \omega_{0n})t/2}{\hbar^2(\omega + \omega_{0n})^2} \right] \tag{1.4}$$

We find a sharp resonance at the Planck-Bohr condition $\omega = \omega_{0n} = (E_n - E_0)/\hbar$. In order to obtain an expression for the probability that varies monotonically with t, we integrate over ω for a small range of frequencies around ω_{0n} and obtain

$$\int_{\text{line}} |C_n(t)|^2 \, d\omega = e^2\mathscr{E}^2(z_{0n})^2 \frac{\pi t}{2\hbar^2}. \tag{1.5}$$

The transition rate $|C_n|^2/t$ is independent of t, as expected.

The cross section σ for photon absorption is defined as

$\sigma =$ (transitions/sec)/photon flux. The cross section integrated over the absorption line

$$\int_{\text{line}} \sigma \, d\omega = \frac{\int_{\text{line}} (\text{transitions/sec}) \, d\omega}{\text{photon flux}}$$

$$= \frac{4\pi^2 e^2 W (z_{0n})^2}{\hbar^2 c}. \tag{1.6}$$

We have used photon energy $W = \hbar\omega$, and photon flux $= c\mathscr{E}^2/8\pi W$.

We now introduce the definition of the oscillator strength f_{0n} for $E1$ transitions between discrete states:

$$f_{0n} = \frac{2mW}{\hbar^2} (z_{0n})^2 = (z_{0n}/\lambda_{0n})^2. \tag{1.7}$$

(Here λ_{0n} is the reduced de Broglie wavelength for a particle of energy W, i.e., $\lambda = \hbar/\text{momentum}$.) Equation 1.6 is rewritten as

$$\int_{\text{line}} \sigma_{0n} \, dW = \frac{2\pi^2 e^2 \hbar}{mc} f_{0n}. \tag{1.8}$$

For transitions to continuum states we remove the integral sign in equation 1.8, and introduce the absorption cross section $\sigma(W)$ and the oscillator density df/dW:

$$\sigma(W) = (2\pi^2 e^2 \hbar/mc) \, df/dW. \tag{1.9}$$

We can also calculate the transition probability $\Gamma_\gamma/\hbar$ to spontaneous photon emission from discrete excited state n (degeneracy g_n) to ground state 0 (degeneracy g_0) using the principle of detailed balance:

$$\Gamma_\gamma/\hbar = \frac{2e^2\omega^2}{mc^3} (g_0/g_n)f_{0n}. \tag{1.10}$$

Equation 1.10 gives the relative width Γ_γ/W for spontaneous emission as

$$\Gamma_\gamma/W = \frac{e^2}{\hbar c} \frac{W}{mc^2} f_{0n}(2g_0/g_n). \tag{1.11}$$

Thus radiation broadening leaves spectrum lines very sharp: $e^2/\hbar c \ll 1$, in general $W \ll mc^2$, and $f_{0n} < 1$.

As we can see from equation 1.7, f_{0n} is a dimensionless quantity. Its order of magnitude is unity for strong electric dipole transitions, since f_{0n} obeys the Thomas-Reiche-Kuhn (TRK) sum-rule [25]

$$\sum_n f_{0n} = 1. \tag{1.12}$$

This equation holds for a single electron system considering $E1$ transitions without retardations. Further, as discussed in section 1.3, the oscillator strength f_{0n} provides connexions between quantum-mechanical and classical theory.

Equation 1.12 is derived using the Heisenberg matrix relations [151]

$$Wz_{0n} = (E_n - E_0)(z_{0n}) = -[H, z]_{0n} = [H, z]_{n0}. \tag{1.13}$$

H is the Hamiltonian operator, and the square brackets denote the commutator. Taking care to keep our quantities Hermitian by using the Heisenberg relation alternately for the first and second z_{0n} of equation 1.7, we find the summed oscillator strength:

$$\sum_n f_{0n} = \frac{2m}{\hbar^2} \sum_n (E_n - E_0) z_{0n} z_{0n}$$

$$= -\frac{m}{\hbar^2} \sum_n \{[H, z]_{0n} z_{0n} + z_{0n}[H, z]_{0n}\}. \tag{1.14}$$

Using closure, we express the summed oscillator strength in terms of properties of the ground (0) state alone:

$$\sum_n f_{0n} = -\frac{m}{\hbar^2} \{[H, z]z - z[H, z]\}_{00}$$

$$= -\frac{m}{\hbar^2} \{[[H, z]z]\}_{00}. \tag{1.15}$$

Substituting $H = \frac{p^2}{2m} + V(r)$,

$$\sum_n f_{0n} = -\frac{m}{\hbar^2} \left\{ \left[\left[\frac{p^2}{2m}, z \right], z \right] \right\}_{00} - \frac{m}{\hbar^2} \{[[V, z], z]\}_{00}. \tag{1.16}$$

The second term is zero if the potential V is a function only of position. Using $[p, z] = -i\hbar$ we can show that the first term is unity. Thus the TRK sum-rule of equation 1.12 holds for *any* one-electron system.

Bethe and Salpeter [25] also work out partial sum-rules (their equations 61.4, 61.5) showing "that a change of principal and orbital quantum numbers in the same sense is more probable than a jump in the opposite sense".

1.3. Correspondence between quantum-mechanical and classical results

The quantum-mechanical TRK sum-rule of equation 1.12 corresponds to the classical integrated power absorption for forced oscillations by a charged oscillator. From equation 1.5 and the TRK rule, we obtain the power absorption integrated over all frequencies for a charged oscillator as $\dfrac{\pi}{4}\dfrac{e^2\mathscr{E}^2}{m}$ for *any* form of $V(r)$ for the oscillator potential. This quantum-mechanical result is in exact agreement with the classical treatment. (See Heitler [98] section 5 for the harmonic oscillator and Van Vleck [200] for the general classical oscillator.) In fact, it was this correspondence between classical and quantum-mechanical results which led Heisenberg to his choice of the value of the commutator $[p, z] = -i\hbar$.

The dispersion relation between the dispersive part of the forward scattering amplitude $F(\omega)$ and the absorption cross section provides another connexion between quantum and classical mechanics. We write the Kramers-Heisenberg dispersion relation for a neutral atom (reference [98] section 19) in the form,

$$F(\omega) = \frac{e^2}{mc^2} \sum_n \frac{\omega^2 f_{0n}}{\omega_{0n}^2 - \omega^2}. \tag{1.17}$$

For an electromagnetic wave of very high frequency ω, we can neglect ω_{0n} in the denominator and equation 1.17 reduces to

$$F(\infty) = -\frac{e^2}{mc^2} \sum_n f_{0n} = -\frac{e^2}{mc^2}. \tag{1.18}$$

Thus dispersion theory and the TRK sum-rule give us the classical Thomson value for the forward scattering amplitude of very high frequency light scattered by an electron.

A third classical interpretation of the oscillator strength is

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found when we compare the quantum-mechanical and classical calculations of the electric polarizability. From quantum-mechanical second order time-independent perturbation theory (reference [151] section 32), we have

$$\alpha = \text{(dipole moment/electric field)}$$

$$= 2e^2 \sum_n \frac{[z_{0n}]^2}{(E_n - E_0)}. \tag{1.19}$$

Expressing $(z_{0n})^2$ in terms of the oscillator strength f_{0n}, by the definition 1.7, we obtain

$$\alpha = e^2 \sum_n \frac{f_{0n}}{m\omega_{0n}{}^2} = e^2 \sum_n \frac{f_{0n}}{k_{0n}}, \tag{1.20}$$

where the "spring constant" $k_{0n} = m\omega_{0n}^2$. This quantum-mechanical result agrees with a classical calculation of α if we interpret the oscillator strength f_{0n} as the fraction of an electron bound by a linear spring of spring constant k_{0n}.

1.4. Dipole sum-rules

In this section, we shall derive sum-rules of the form $\sum_n W^p f_{0n}$ for a one-electron system [19, 25]. Here W is the photon energy, and $p = \pm 1,\ \pm 2,\ 3$. The TRK sum-rule equation 1.12 is the special case $p = 0$.

We can use these sum-rules to calculate different mean energies for the photo-effect. We define the pth statistical moment μ_p and the pth mean energy W_p as follows:

$$\mu_p = \int_0^\infty W^p \frac{df}{dW}\, dW. \tag{1.21}$$

$$W_p = (\mu_p/\mu_0)^{1/p} = (\mu_p)^{1/p}. \tag{1.22}$$

We have used $\mu_0 = 1$ for a one-electron system for electric dipole transitions. df/dW is the oscillator strength per unit energy. (Here we integrate over the continuum of photon energies, while previously we summed over discrete photon energies. The two notations are equivalent.)

Since larger values of p weight higher values of W more strongly, we have

$$W_3 \geq W_2 \geq W_1 \geq W_{-1} \geq W_{-2}, \qquad (1.23)$$

the equalities holding only if df/dW has a single delta function peak.

The sum-rule for W_{-2} involves the calculation of the polarizability α discussed in section 1.3. From equations 1.20 and the definitions 1.21 and 1.22,

$$\mu_{-2} = \int \frac{\dfrac{df}{dW} \, dW}{W^2} = \frac{m\alpha}{e^2 \hbar^2}; \quad W_{-2} = (1/\mu_{-2})^{1/2}. \qquad (1.24)$$

We calculate μ_{-1} directly from the definition, using closure.

$$\mu_{-1} = \sum_n \frac{f_{0n}}{W} = \frac{2m}{\hbar^2} \sum_n (z_{0n})^2 = \frac{2m}{\hbar^2} (z^2)_{00} = \frac{2}{3} \frac{m}{\hbar^2} (r^2)_{00}$$

$$1/\mu_{-1} = W_{-1} = \frac{3\hbar^2}{2m} \frac{1}{(r^2)_{00}}. \qquad (1.25)$$

The harmonic mean energy W_{-1} is inversely proportional to the mean square radius $(r^2)_{00}$ [127].

We calculate μ_1 using the matrix relation of section 1.2:

$$(E_n - E_0) z_{0n} = \frac{i\hbar}{m} (p_z)_{0n}. \qquad (1.26)$$

$$\mu_1 = \sum_n W f_{0n} = \frac{2m}{\hbar^2} \sum_n (E_n - E_0)^2 z_{0n} z_{0n}$$

$$= \frac{2m}{\hbar^2} \frac{\hbar^2}{m^2} \sum_n (p_z)_{0n} (p_z)_{n0} = \frac{2}{m} (p_z^2)_{00} = \frac{2}{3m} (p^2)_{00} = \frac{4}{3} (T)_{00}, \qquad (1.27)$$

where T_{00} is the expectation value of the kinetic energy.

In the one-electron case $\mu_1 = W_1$. W_1 is called the mean energy, and is proportional to the mean kinetic energy.

We calculate μ_2 using equation 1.26 and another matrix relation corresponding to the classical relation that the acceleration is proportional to the force:

$$(E_n - E_0)^2 z_{0n} = \frac{\hbar^2}{m} \left(\frac{\partial V}{\partial z} \right)_{0n}; \qquad (1.28)$$

$$\mu_2 = \sum_n W^2 f_{0n} = \frac{2m}{\hbar^2} \sum_n (E_n - E_0)^3 z_{0n} z_{0n}$$

$$= \frac{2m}{\hbar^2} \frac{\hbar^3}{m^2} \frac{1}{2} \sum_n \left\{ \left(\frac{\partial V}{\partial z}\right)_{0n} (p_z)_{0n} + (p_z)_{0n} \left(\frac{\partial V}{\partial z}\right)_{0n} \right\} \quad (1.29)$$

$$= \frac{\hbar^2}{m} \left(\frac{\partial^2 V}{\partial z^2}\right)_{00} = \frac{\hbar^2}{3m} (\nabla^2 V)_{00},$$

$$W_2 = \left[\frac{\hbar^2}{3m} (\nabla^2 V)_{00} \right]^{1/2}. \quad (1.30)$$

We find W_3 and μ_3 using the matrix relation 1.28 twice, giving us

$$\mu_3 = \frac{2\hbar^2}{m} \left[\left(\frac{\partial V}{\partial z}\right)^2 \right]_{00} \quad (1.31)$$

and

$$W_3 = \left\{ \frac{2\hbar^2}{m} \left[\left(\frac{\partial V}{\partial z}\right)^2 \right]_{00} \right\}^{1/3}. \quad (1.32)$$

We can use the above sum-rules to calculate the various mean energies for photon absorption by the hydrogen atom. Our work is only illustrative since the values of f_{0n} for discrete transitions and df/dW for the continuum (the Stobbe formula) are known explicitly [25, 193]. But the sum-rule method, where we need only the ground state wave function can be used for other problems, where explicit calculations necessitating knowledge of wave functions for excited states may be extremely difficult.

The electric polarizability α of hydrogen can be calculated analytically using parabolic coordinates (reference [151], section 32) giving $\alpha = \frac{9}{2} a_0^3$, which by equation 1.24 gives

$$W_{-2} = \left(\frac{8}{9}\right)^{1/2} \text{Ry}. \quad \left(\text{Here } a_0 \text{ is the Bohr radius and } 1 \text{ Ry} = \frac{e^2}{2a_0}. \right)$$

Since $(r^2)_{00} = 3a_0^2$, we find $W_{-1} = 1$ Ry. Since $(T)_{00} = 1$ Ry, $W_1 = \frac{4}{3}$ Ry. Finally $(\nabla^2 V)_{00} = 4e^2/a_0^2$, giving $W_2 = (16/3)^{1/2}$

Ry. μ_3 and W_3 are infinite because $\left(\dfrac{\partial V}{\partial z}\right)^2$ has an r^{-4} singularity at the origin. Thus equation 1.23 is satisfied: $\infty \geq (16/3)^{1/2} \geq 4/3 \geq 1 \geq (8/9)^{1/2}$ for the various mean energies.

If the asymptotic form of df/dW at high energies is W^{-n}, the above argument shows that $3 < n \leq 4$ for hydrogen. (The Stobbe formula gives $n = 7/2$.) In general the value of n depends only on the form of the singularity in the potential $V(r)$ [127, 180, 217].

We can also use these sum-rules to show a relation between two well-known properties of the simple harmonic potential. [131] For any distribution of oscillator strengths, we must have $W_1 \geq W_{-1}$. Using equations 1.25 and 1.27, we find

$$\frac{2}{m}(p_z^2)_{00} \geq \frac{\hbar^2}{2m}\frac{1}{(z^2)_{00}}$$

or   (1.33)

$$(p_z^2)_{00}(z^2)_{00} \geq \frac{\hbar^2}{4},$$

which is in agreement with the Heisenberg uncertainty principle. The equality $W_1 = W_{-1}$ holds only if df/dW is a Dirac delta function, as it indeed is for the harmonic oscillator. Thus the equality in the Heisenberg principle and the single delta function peak in the photo-effect both hold *only* for the ground state of a simple harmonic oscillator.

The inequality in equation 1.33 is usually not very great for other potentials, for example, for the Coulomb potential discussed above or for a one-dimensional, infinitely deep, square well. For the latter

$$(p_z^2)_{00}(z^2)_{00} = \frac{\pi^2-6}{3}\frac{\hbar^2}{4} = 1{\cdot}230\,\frac{\hbar^2}{4}. \qquad (1.34)$$

Then W_1 is not much greater than W_{-1}, showing that the oscillator density has a rather sharp peak. $\left(\text{Compare the hydrogen atom case where } W_1 = \dfrac{4}{3}\,W_{-1}.\right)$