THE MATHEMATICS OF PHYSICS AND CHEMISTRY

Henry Margenau
George M. Murphy

VOLUME II

THE MATHEMATICS OF PHYSICS AND CHEMISTRY

VOLUME TWO

BY

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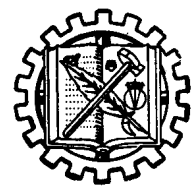

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PREFACE

This book has been conceived as a companion volume of *The Mathematics of Physics And Chemistry*. Two decades have passed since the publication of that book, and during this interval the demands for mathematical knowledge laid by the physical sciences upon their students have both shifted and increased. The early book has become incomplete in its offerings for the student of today, and we have sought to remove this fault.

Unquestionably the best way to achieve that end would have been to rewrite the book, to expand its size and its scope, along some lines which are still unconventional but hold out promise for future interest. The prospect of undertaking this frightened us. Also, we acknowledge that there already exists a monumental treatise of this elaborate sort, dedicated to the standard parts of the newer mathematics, in the work of Morse and Feshbach, whose excellence would be hard to approach. We, therefore, decided upon the less ambitious course of editing a work which offers what we regard as the most important components of today's useful mathematics in separate chapters written chiefly by experts.

We do not underestimate the difficulties inherent in this task. The heterogeneity of highly composite books, in spite of the present vogue which spawns them, always impairs their usefulness and certainly detracts from their teachability. Nor is there a way of avoiding this difficulty. One confronts here the principle of complementarity for editorial surveillance. Written in the form of an uncertainty relation, that principle reads

$$\Delta E \cdot \Delta T \geqq H$$

where ΔE is the uncertainty in the esteem an editor enjoys among his contributors, ΔT a measure of the lack of uniformity of treatment, style and symbolism in different parts of the book, and H is a great deal larger than Planck's constant. (In general H is not a constant, but a function of the contributors' prestige, ranging from small values for modest writers almost to infinity for prima donnas.) If the reader understands these facts he will have the correct perspective on the present volume.

The preparation of a collaborative work such as this requires more human understanding, generous handling and skillful strategy on the part of the publisher than any book of single authorship. We were most fortunate in having a monitor who displayed an abundance of all these qualities in Mr. W. Minrath, Vice President of the Van Nostrand Company, and we want to thank him above all for his unfailing help and his good will.

HENRY MARGENAU
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CONTENTS

CHAPTER 1

TRANSPORT THEORY OF GASES

by

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1.1. Introduction.—Statistical physics deals with the relation between the macroscopic laws that describe the internal state of a system and the dynamics of the interactions of its microscopic constituents. The derivation of the nonequilibrium macroscopic laws, such as those of hydrodynamics, from the microscopic laws has not been developed as generally as in the equilibrium case (the derivation of thermodynamic relations by equilibrium statistical mechanics). The microscopic analysis of nonequilibrium phenomena, however, has achieved a considerable degree of success for the particular case of dilute gases. In this case, the *kinetic theory*, or *transport theory*, allows one to relate the transport of matter or of energy, for example (as in diffusion, or heat flow, respectively), to the mechanics of the molecules that make up the system.

In kinetic theory, the macroscopic quantities are found as averages over the motion of many molecules; each molecular event is assumed to take place over a microscopic time interval, so that a measurement that is made over a macroscopic time interval involves many molecules. The kinetic description is, therefore, a probabilistic one in that assumptions are made about the motion of one molecule and the results of this motion are averaged over all of the molecules of the gas, giving proper weight to the probability that the various molecules of the gas can have the assumed motion.

In its most elementary aspects, kinetic theory is developed on the basis of a hard sphere model of the particles (atoms or molecules) making up the gas.[1] The assumption is made that the particles are uniformly distributed in space and that all have the same speed, but that there are equal numbers of particles moving parallel to each coordinate axis. This last assumption allows one to take averages over

[1] J. O. Hirschfelder, C. F. Curtiss, and R. B. Bird, *Molecular Theory of Gases and Liquids*, pp. 8 ff., John Wiley and Sons, Inc., New York, 1954.

the direction of motion of the particles, and thus, to obtain a statistical description of gross effects. This simple model is found to give results that are reasonable approximations to the macroscopic laws they attempt to describe. Aside from being unrealistic conceptually, however, the very simplicity of the model prevents calculation of many phenomena of interest.

In its more advanced aspects, kinetic theory is based upon a description of the gas in terms of the probability of a particle having certain values of coordinates and velocity, at a given time. Particle interactions are developed by the ordinary laws of mechanics, and the results of these are averaged over the probability distribution. The probability distribution function that is used for a given macroscopic physical situation is determined by means of an equation, the *Boltzmann transport equation*, which describes the space, velocity, and time changes of the distribution function in terms of collisions between particles. This equation is usually solved to give the distribution function in terms of certain macroscopic functions; thus, the macroscopic conditions imposed upon the gas are taken into account in the probability function description of the microscopic situation.

1.2. Distribution Function.—Let us denote a point in space, having rectangular coordinates (x,y,z), by $\mathbf{r}$; the differential volume element $dxdydz$ will be represented by $d\mathbf{r}$. Similarly, the velocity (or point in velocity space) $\mathbf{v}$ will have rectangular components (v_x,v_y,v_z); the volume element in velocity space, $dv_xdv_ydv_z$, will be represented by $d\mathbf{v}$. If dN is the number of particles which are in the differential volume $d\mathbf{r}$, at $\mathbf{r}$, and have their velocities in the range $d\mathbf{v}$, at $\mathbf{v}$, then the *distribution function* is defined by:

$$dN = f(\mathbf{r},\mathbf{v},t)d\mathbf{r}d\mathbf{v} \tag{1-1}$$

The differential lengths and velocities considered must be small compared with the macroscopic distances and velocity intervals over which there are significant changes in the gross properties of the gas. On the other hand, they must be sufficiently large so that there are a large number of particles contained in the differential space-velocity volume; this allows $f(\mathbf{r},\mathbf{v},t)$ to be a continuous function of its variables.[2]

If Eq. (1–1) is integrated over all of velocity space ($-\infty \leq v_x,v_y,v_z \leq \infty$, in general), then the number of particles in the volume $d\mathbf{r}$, at $\mathbf{r}$,

[2] The definition of the distribution function can best be made in terms of an ensemble average (see H. Margenau and G. M. Murphy, *The Mathematics of Physics and Chemistry*, 2nd Ed., p. 442, D. Van Nostrand Co., Inc., Princeton, N.J., 1956), and a connection can be made with the Liouville theorem of statistical mechanics. Cf. H. Grad, "Principles of the Kinetic Theory of Gases," *Handbuch der Physik*, Vol. XII, pp. 206 ff., Springer Verlag, Berlin, 1958.

is obtained; this gives the number of particles per unit volume, or the number density, as:

$$n(\mathbf{r},t) = \int f(\mathbf{r},\mathbf{v},t)d\mathbf{v} \tag{1-2}$$

For a gas consisting of only one constituent, of mass m, the mass density is

$$\rho(\mathbf{r},t) = mn(\mathbf{r},t) \tag{1-3}$$

1.3. Two-Particle Collisions.—One of the basic assumptions in the derivation of the Boltzmann equation is that the gas being described is sufficiently dilute so that only two-particle collisions are of importance. The mechanics of a two-body encounter will thus be described in order

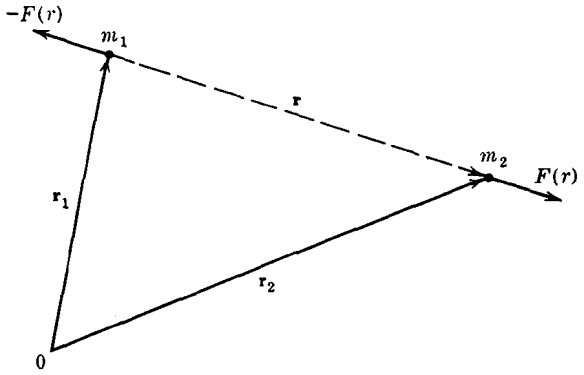

FIG. 1–1. The Two-Particle Encounter.

to derive a relation for the angle of deflection of the particles from their paths before the encounter, in terms of the relative velocity of the particles and their impact parameter, which will be defined shortly.

It is assumed that the particles, which have no internal degrees of freedom, interact through a central force field, that is, the interparticle force depends only upon the distance between the particles, and acts along the line connecting them. Since we are restricting the system to collisions between only two particles at a time, the force is assumed to have a finite range. To a good degree of approximation, this means that the force decreases with distance sufficiently rapidly so that it has negligible effect on the motion at distances of the order of the interparticle spacing ($\sim n^{-1/3}$). If the particles, of mass m_1 and m_2, are at

the positions $\mathbf{r}_1$ and $\mathbf{r}_2$ with respect to some origin, the equations of motion are (see Fig. 1–1):

$$m_1\ddot{\mathbf{r}}_1 = -F(r)\mathbf{i}_r; \qquad m_2\ddot{\mathbf{r}}_2 = F(r)\mathbf{i}_r$$
$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1; \qquad r = |\mathbf{r}|; \qquad \mathbf{i}_r = \mathbf{r}/|\mathbf{r}| \tag{1–4}$$

where $F(r)$ is the magnitude of the force on particle two due to particle one. Any external force acting on the particles is assumed not to vary over the distances or times involved in the collision, so that the relative motion of the particles is unaffected by the external force. The effect of the external force will be assumed to be small, compared with the interaction forces during a collision, and will be neglected here; thus, the center of mass of the system moves with constant velocity. The plane formed by the relative velocity of the particles and the line joining their position will contain the relative position vector of the particles for all time.[3] The relative motion of the particles may be obtained from Eq. (1–4):

$$\mu\ddot{\mathbf{r}} = F(r)\mathbf{i}_r; \qquad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \tag{1–5}$$

This describes the motion of a single particle having the reduced mass[4] of the two-particle system, whose position is that of particle two with respect to particle one, and which is acted upon by the interparticle force.

Let the *initial* velocities of the particles be $\mathbf{v}_1$ and $\mathbf{v}_2$, and their *final* velocities be $\mathbf{v}_1'$ and $\mathbf{v}_2'$; the initial and final relative velocities will be

$$\mathbf{g} = \mathbf{v}_2 - \mathbf{v}_1; \qquad \mathbf{g}' = \mathbf{v}_2' - \mathbf{v}_1' \tag{1–6}$$

The velocity of the center of mass G is the same before and after the collision:

$$\mathbf{G} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2} = \mathbf{G}' \tag{1–7}$$

If, as in Fig. 1–2, we describe the relative motion in the plane by means of polar coordinates (r,θ), conservation of energy requires:

$$\tfrac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = \tfrac{1}{2}\mu g^2 = \tfrac{1}{2}\mu g'^2;$$
$$g = |\mathbf{g}|; \qquad g' = |\mathbf{g}'| \tag{1–8}$$

[3] R. B. Lindsay, *Concepts and Methods of Theoretical Physics*, pp. 73–4, D. Van Nostrand Co., Inc., Princeton, N.J., 1951.

[4] H. Margenau and G. M. Murphy, *The Mathematics of Physics and Chemistry*, p. 413, D. Van Nostrand Co., Inc., Princeton, N.J., 1956.

where $V(r)$ is the potential associated with the interparticle force: $F(r) = -\partial V/\partial r$. Here, the total energy has been set equal to the initial kinetic energy, and to the final kinetic energy, since, by the assumption of a finite range of force, V is zero at the beginning and at the end of a collision. We see, therefore, that the relative velocity after collision is equal to the relative velocity before collision rotated through the angle χ of Fig. 1–2.

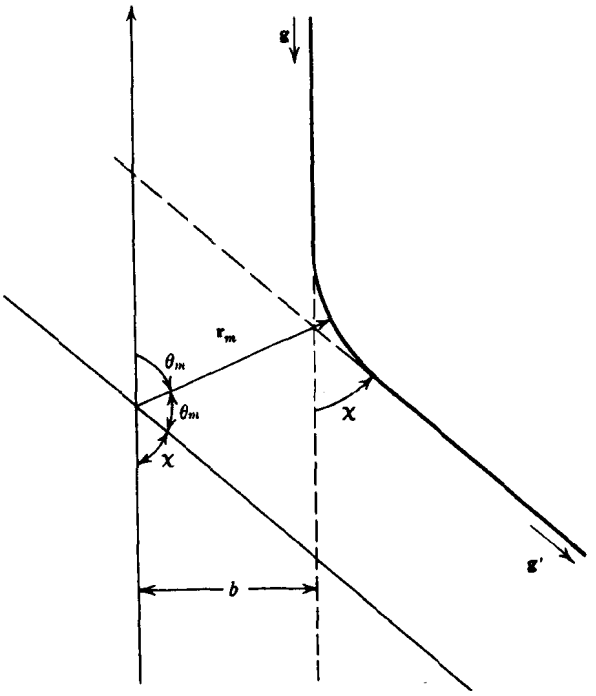

FIG. 1–2. Geometry of a Collision in a Central Force Field.

The impact parameter, b, is defined to be the perpendicular distance between the initial relative path (along $\mathbf{g}$) and the line parallel to $\mathbf{g}$ through the force center (b would be the distance of closest approach of the particles, if there were no interaction); the initial angular momentum is just μbg. Conservation of angular momentum [5] is thus:

$$\mu r^2\dot{\theta} = \mu bg = \mu bg' \tag{1-9}$$

where $\mu bg'$ is the final angular momentum [note that $g' = g$ from

[5] R. B. Lindsay, *loc. cit.*

Eq. (1–8)]. The orbit is symmetric about a line through the force center that bisects the angle between the initial and final relative paths (the line $\mathbf{r}_m$ of Fig. 1–2).

The distance of closest approach, r_m, is obtained by using the first of Eq. (1–9) in the first of Eq. (1–8), and setting $\dot{r} = 0$. Thus we find:

$$\frac{b^2}{r_m^2} = 1 - \frac{V(r_m)}{\frac{1}{2}\mu g^2} \tag{1–10}$$

This determines the distance of closest approach in terms of the initial relative velocity, the impact parameter, and the dynamical quantities (masses and force law constants). The equation for the orbit of the relative motion is found from the first of Eqs. (1–8) and (1–9), using the identity $(\dot{r}/\dot{\theta}) = (dr/d\theta)$, as follows:

$$\left(\frac{dr}{d\theta}\right)^2 + r^2 = \left(\frac{r^4}{b^2 g^2}\right)\left[g^2 - \frac{2}{\mu} V(r)\right]$$

The angle at which the minimum distance of approach occurs, θ_m (see Fig. 1–2), is thus:

$$\theta_m = \int_0^{\theta_m} d\theta = -\int_\infty^{r_m} (d\theta/dr)dr$$

$$\therefore\ \theta_m = -\int_\infty^{r_m} \frac{b\,dr}{r^2[1 - (b^2/r^2) - (2/\mu g^2)V(r)]^{1/2}} \tag{1–11}$$

From Fig. 1–2, the angle of deflection, χ, is given by:

$$\chi = \pi - 2\theta_m \tag{1–12}$$

Thus, from Eqs. (1–10), (1–11), and (1–12), the angle of deflection can be found in terms of the parameters describing the collision (g, b, μ, and the force constants). For the two-body collision, therefore, the final velocities of the particles, $\mathbf{v}_1'$ and $\mathbf{v}_2'$, are functions of the initial velocities $\mathbf{v}_1$ and $\mathbf{v}_2$, and the above parameters.

1.4. Angle of Deflection for Some Simple Cases.—If, as is often assumed for simplicity, the interparticle force law is given by

$$F(r) = \frac{K}{r^\nu}, \qquad \nu \geq 3 \tag{1–13}$$

the equations may be written in a simple form. Defining:

$$\beta = b/r; \qquad \beta_0 = b/r_m; \qquad b_0 = b(\mu g^2/K)^{1/(\nu-1)} \tag{1–14}$$

we have, from Eq. (1–10):

$$1 - \beta_0^2 - \frac{2}{\nu - 1}(\beta_0/b_0)^{\nu-1} = 0 \tag{1–15}$$

and, from Eqs. (1–11) and (1–12)

$$\chi(b_0) = \pi - 2\int_0^{\beta_0} \left[1 - \beta^2 - \frac{2}{\nu - 1}(\beta/b_0)^{\nu-1}\right]^{-1/2} d\beta \qquad (1\text{–}16)$$

One determines β_0 from Eq. (1–15), and then uses it in the integral of Eq. (1–16). For this inverse power law, the angle of deflection is thus dependent upon all of the parameters of the collision only through the single parameter b_0, and ν.

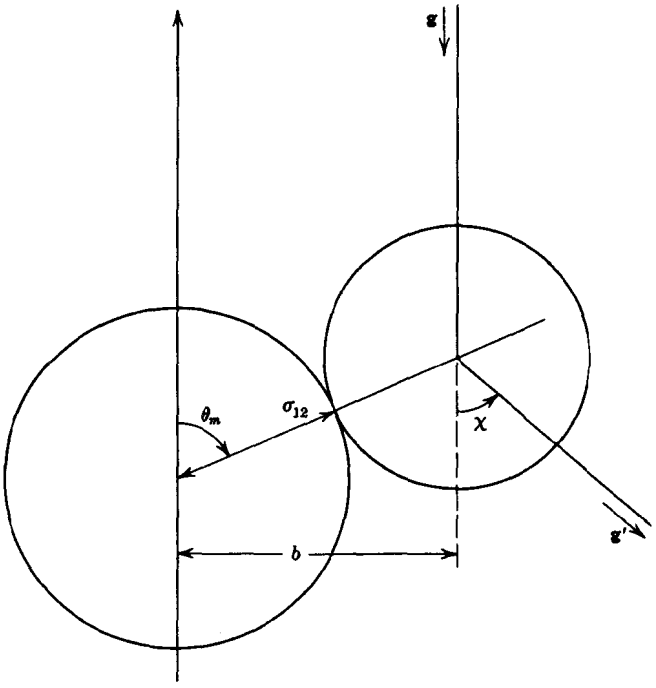

FIG. 1–3. Geometry of a Collision for Hard Spheres.

The angle of deflection for the collision of rigid elastic spheres may be obtained from Fig. 1–3. The minimum distance of approach is

$$r_m = \sigma_{12} = \tfrac{1}{2}(\sigma_1 + \sigma_2) \qquad (1\text{–}17)$$

where σ_1, σ_2 are the diameters of the two colliding particles. From the geometry, we see that

$$b = \sigma_{12} \sin \theta_m$$

so that the angle of deflection is given by:

$$\chi = \begin{cases} \pi - 2 \sin^{-1} (b/\sigma_{12}) & b \leq \sigma_{12} \\ 0 & b \geq \sigma_{12} \end{cases} \tag{1-18}$$

1.5. Some Vector Relations.—Since the relative motion takes place in a plane, one further parameter must be given in order to describe the collision in three dimensions, namely the orientation of the relative-motion plane. This may be done as in Fig. 1–4, where the collision

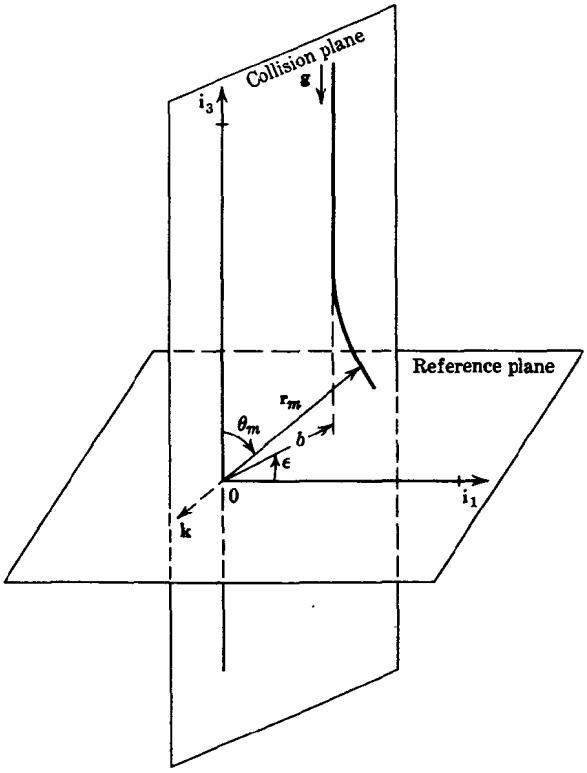

FIG. 1–4. Three-Dimensional Geometry of a Collision.

plane is shown, and a reference plane is drawn through the origin in the collision plane (the origin being the location of the force center) and perpendicular to the relative velocity vector $\mathbf{g}$. The angle ε, measured from an arbitrary line in the reference plane to the collision plane, then defines the location of the latter. For a given value of

relative velocity, the parameters b and ε specify the collision; since the angle of deflection may be found from the previous equations in terms of the geometrical factor b, the angles χ, ε may also be used to specify the collision.

If the unit vector $\mathbf{k}$ is chosen along $\mathbf{r}_m$, but directed opposite to it, $\mathbf{k}$ can be represented in terms of the angles $\pi - \theta_m$ and $\pi + \varepsilon$, or $\frac{1}{2}(\pi + \chi)$ and $\pi + \varepsilon$, where the angle θ_m is measured from the polar axis which is antiparallel to $\mathbf{g}$, as in Fig. 1–3. Noting that χ is the angle between $\mathbf{g}$ and $\mathbf{g}'$, and that $\mathbf{g}$ and $\mathbf{k}$ form an angle θ_m, the vector triangle of Fig. 1–5 can be drawn; here, the magnitude a is a size parameter, since $\mathbf{k}$ has been chosen of unit length. From the relation between χ and θ_m, it can be seen that the triangle is isosceles, so that

$$a = 2g \sin(\chi/2) = 2g \cos\theta_m \equiv 2\mathbf{g}\cdot\mathbf{k}$$

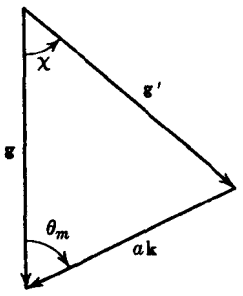

FIG. 1–5. Vector Triangle for a Collision.

Therefore

$$\mathbf{g} = \mathbf{g}' + 2(\mathbf{g}\cdot\mathbf{k})\mathbf{k} \qquad (1\text{–}19)$$

From Eqs. (1–6) and (1–7), we obtain:

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{G} - \left(\frac{m_2}{m_1 + m_2}\right)\mathbf{g}; \qquad \mathbf{v}_2 = \mathbf{G} + \left(\frac{m_1}{m_1 + m_2}\right)\mathbf{g} \\ \mathbf{v}_1' &= \mathbf{G} - \left(\frac{m_2}{m_1 + m_2}\right)\mathbf{g}'; \qquad \mathbf{v}_2' = \mathbf{G} + \left(\frac{m_1}{m_1 + m_2}\right)\mathbf{g}' \end{aligned} \qquad (1\text{–}20)$$

If we use Eq. (1–19) in the above relations, the changes in velocity of each of the colliding particles are found to be:

$$\begin{aligned} \mathbf{v}_1' - \mathbf{v}_1 &= 2\left(\frac{m_2}{m_1 + m_2}\right)(\mathbf{g}\cdot\mathbf{k})\mathbf{k} \\ \mathbf{v}_2' - \mathbf{v}_2 &= -2\left(\frac{m_1}{m_1 + m_2}\right)(\mathbf{g}\cdot\mathbf{k})\mathbf{k} \end{aligned} \qquad (1\text{–}21)$$

From these relations between the initial and final velocities of each of the particles, it follows that the jacobian of the transformation is unity, so that

$$d\mathbf{v}_1' d\mathbf{v}_2' = d\mathbf{v}_1 d\mathbf{v}_2 \qquad (1\text{–}22)$$

The velocity vectors of the particles are usually represented in some stationary macroscopic coordinate system; it is this coordinate system that is used to describe the position vector $\mathbf{r}$ of the distribution

function. However, the integrations involved in the integrals of the later sections are performed in terms of the relative velocity coordinate system and the center of mass coordinate system. The transformations of Eqs. (1–20) and (1–21) are used, together with the total energy of the system which can be found directly from Eq. (1–20):

$$\tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 = \tfrac{1}{2}(m_1 + m_2)G^2 + \tfrac{1}{2}\mu g^2 \qquad (1\text{–}23)$$

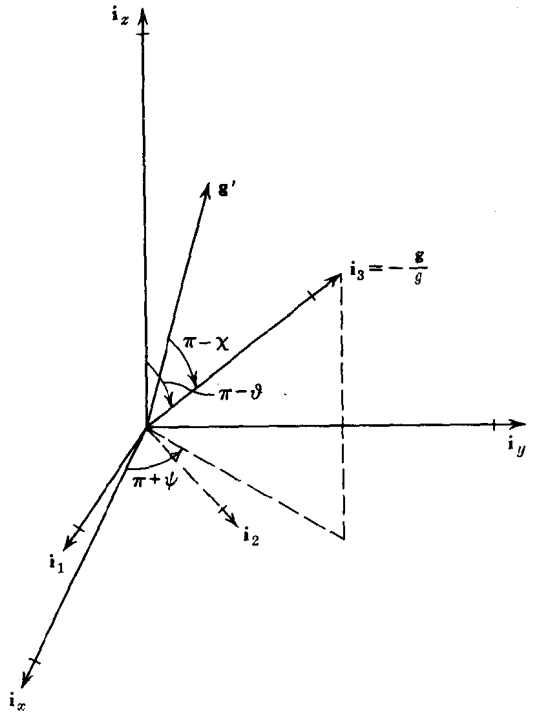

FIG. 1–6. Gas Coordinate System and Relative Velocity Coordinate System.

The transformations relating $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{G}$, $\mathbf{g}$ are Eqs. (1–20), and are such that

$$d\mathbf{v}_1 d\mathbf{v}_2 = d\mathbf{G} d\mathbf{g} \qquad (1\text{–}24)$$

In the relative velocity system of Fig. 1–4, a rectangular coordinate system is set up as follows: the $\mathbf{i}_3$ axis is the line antiparallel to $\mathbf{g}$, through the origin; the $\mathbf{i}_1$ axis is the line in the reference plane from which ϵ is measured; the $\mathbf{i}_2$ axis is in the reference plane, perpendicular

to $\mathbf{i}_1$ and $\mathbf{i}_3$, forming a right-handed system. The coordinates of $\mathbf{g}$ are then

$$\mathbf{g} = (0,0,-g) \tag{1-25}$$

the unit vector $\mathbf{k}$, at angles $\frac{1}{2}(\pi + \chi)$ and $\pi + \varepsilon$ has rectangular coordinates:

$$\mathbf{k} = (-\cos(\chi/2)\cos\varepsilon, \ -\cos(\chi/2)\sin\varepsilon, \ -\sin(\chi/2)) \tag{1-26}$$

the vector $\mathbf{g}'$ is found from Eq. (1–19) to be:

$$\mathbf{g}' = (g\sin\chi\cos\varepsilon, \ g\sin\chi\sin\varepsilon, \ -g\cos\chi) \tag{1-27}$$

These representations will be used to calculate collision integrals in Section 1.15.

Since the vector $\mathbf{g}'$ is represented above in terms of the $\mathbf{g}$-coordinate system $(\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3)$ having $-\mathbf{g}$ as the $\mathbf{i}_3$ axis, it is necessary to determine the transformation to the $(\mathbf{i}_x,\mathbf{i}_y,\mathbf{i}_z)$ coordinate system in which the particle velocities are written, in order to evaluate certain integrals. If we let (ϑ,ψ) be the spherical coordinate angles of the vector $\mathbf{v}_2 - \mathbf{v}_1$, in the $\mathbf{v}$-coordinate system, then:

$$\begin{aligned}\mathbf{g} &= (v_{2x} - v_{1x})\mathbf{i}_x + (v_{2y} - v_{1y})\mathbf{i}_y + (v_{2z} - v_{1z})\mathbf{i}_z \\ &= g[\sin\vartheta\cos\psi\mathbf{i}_x + \sin\vartheta\sin\psi\mathbf{i}_y + \cos\vartheta\mathbf{i}_z]\end{aligned} \tag{1-28}$$

The unit vector $\mathbf{i}_3$ extends along $-\mathbf{g}$ as already mentioned (see Fig. 1–6); the $(\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3)$ coordinate axes may thus be written in terms of the $(\mathbf{i}_x,\mathbf{i}_y,\mathbf{i}_z)$ axes as follows:

$$\begin{aligned}\mathbf{i}_3 &= -[\sin\vartheta\cos\psi\mathbf{i}_x + \sin\vartheta\sin\psi\mathbf{i}_y + \cos\vartheta\mathbf{i}_z] \\ \mathbf{i}_1 &= -\sin\psi\mathbf{i}_x + \cos\psi\mathbf{i}_y \\ \mathbf{i}_2 &= \cos\vartheta\cos\psi\mathbf{i}_x + \cos\vartheta\sin\psi\mathbf{i}_y - \sin\vartheta\mathbf{i}_z\end{aligned} \tag{1-29}$$

The $\mathbf{i}_1$ axis has been chosen in the $\mathbf{i}_x$-$\mathbf{i}_y$ plane; this choice is arbitrary and serves to define the axis from which the polar "scattering" angle ε is measured. Thus we have:

$$\mathbf{g} = -g\mathbf{i}_3 \tag{1-30}$$

$$\begin{aligned}\mathbf{g}' = g[&(\sin\vartheta\cos\psi\cos\chi + \cos\vartheta\cos\psi\sin\chi\sin\varepsilon - \sin\psi\sin\chi\cos\varepsilon)\mathbf{i}_x \\ &+ (\sin\vartheta\sin\psi\cos\chi + \cos\vartheta\sin\psi\sin\chi\sin\varepsilon + \cos\psi\sin\chi\cos\varepsilon)\mathbf{i}_y \\ &+ (\cos\vartheta\cos\chi - \sin\vartheta\sin\chi\sin\varepsilon)\mathbf{i}_z]\end{aligned} \tag{1-31}$$

1.6. Inverse Collisions.—The particle velocities resulting from a collision between particles of velocities $\mathbf{v}_1$ and $\mathbf{v}_2$, having collision parameters b and ε, have been denoted as $\mathbf{v}_1'$ and $\mathbf{v}_2'$; they may be found from Eqs. (1–21). Consider now the particle velocities resulting from a collision between particles of velocities $\mathbf{v}_1'$ and $\mathbf{v}_2'$, with collision parameters b and ε; let these final velocities be denoted by $\mathbf{v}_1''$ and $\mathbf{v}_2''$.

The collision that takes $(\mathbf{v}_1,\mathbf{v}_2)$ into $(\mathbf{v}_1',\mathbf{v}_2')$ will be called the direct collision; that that takes $(\mathbf{v}_1',\mathbf{v}_2')$ into $(\mathbf{v}_1'',\mathbf{v}_2'')$ will be called the inverse collision; see Fig. 1–7. Equations (1–9) and (1–10), the conservation laws for energy and for angular momentum, applied to the new system, yield $g'' = g'$; since it was found that, for the original system, $g' = g$,

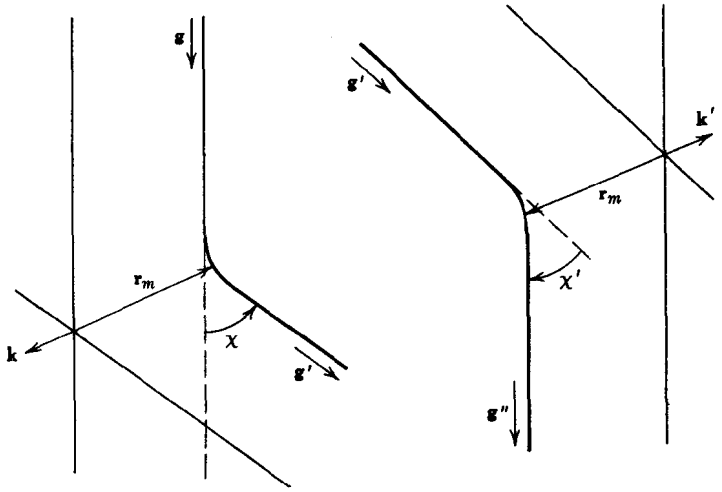

FIG. 1–7. Direct and Inverse Collisions.

we see that the final relative speed for the inverse collision is the same as the initial relative speed for the direct collision:

$$g'' = g$$

Since the orbit depends only upon g, b and the dynamical quantities (masses and force law), the angle of deflection in the inverse collision is the same as that for the direct collision (cf. Eq. (1–11)); however, the unit vector $\mathbf{k}'$ for the inverse collision is the negative of that for the direct collision. If, in Eq. (1–21), $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_1'$, $\mathbf{v}_2'$, $\mathbf{g}$, $\mathbf{k}$ are replaced, respectively, by $\mathbf{v}_1'$, $\mathbf{v}_2'$, $\mathbf{v}_1''$, $\mathbf{v}_2''$, $\mathbf{g}'$, $-\mathbf{k}$, these equations become:

$$\mathbf{v}_1'' - \mathbf{v}_1' = +2\left(\frac{m_2}{m_1 + m_2}\right)(\mathbf{g}'\cdot\mathbf{k})\mathbf{k}$$

$$\mathbf{v}_2'' - \mathbf{v}_2' = -2\left(\frac{m_1}{m_1 + m_2}\right)(\mathbf{g}'\cdot\mathbf{k})\mathbf{k}$$

Noting, from Fig. 1–5, that $\mathbf{g}'\cdot\mathbf{k} = -(\mathbf{g}\cdot\mathbf{k})$, these equations reduce to Eq. (1–21), with

$$\mathbf{v}_1'' = \mathbf{v}_1; \qquad \mathbf{v}_2'' = \mathbf{v}_2$$