Graduate Texts in Mathematics

Arlen Brown
Carl Pearcy

Introduction to Operator Theory I

Elements of Functional Analysis

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For Dodie and Cristina,
who have given constant encouragement
over many years

Preface

This book was written expressly to serve as a textbook for a one- or two-semester introductory graduate course in functional analysis. Its (soon to be published) companion volume, *Operators on Hilbert Space*, is intended to be used as a textbook for a subsequent course in operator theory. In writing these books we have naturally been concerned with the level of preparation of the potential reader, and, roughly speaking, we suppose him to be familiar with the approximate equivalent of a one-semester course in each of the following areas: linear algebra, general topology, complex analysis, and measure theory. Experience has taught us, however, that such a sequence of courses inevitably fails to treat certain topics that are important in the study of functional analysis and operator theory. For example, tensor products are frequently not discussed in a first course in linear algebra. Likewise for the topics of convergence of nets and the Baire category theorem in a course in topology, and the connections between measure and topology in a course in measure theory. For this reason we have chosen to devote the first ten chapters of this volume (entitled Part I) to topics of a preliminary nature. In other words, Part I summarizes in considerable detail what a student should (and eventually must) know in order to study functional analysis and operator theory successfully. The presence of this extensive review of the prerequisite material means that a student who is not familiar with one or more of the four basic courses mentioned above may still successfully read this book by making liberal use of Part I. Indeed, it should be said that perhaps the only critical prerequisite for a profitable reading of this book is a certain mathematical maturity, which, for our purposes, may be taken to mean the ability to follow and construct ε-δ arguments, a level of maturity that any talented student who has had a good course in advanced calculus will have attained.

In keeping with our pedagogical intent in writing this book, we have provided both examples and exercises in copious supply. Indeed, every chapter contains a number of illuminating examples and is followed by a collection of problems. (Some problems appear as simple assertions of fact; in such cases the student is expected to provide a proof of the stated fact.) In this connection we observe that the problem sets constitute an integral part of the book, and that the student must study them along with the text. Working problems is very important in the study of mathematics in general, of course, for that is how mathematics is learned, but in this textbook it is particularly important because many topics of interest are first introduced in the problems. Not infrequently the solution of a problem depends in part on material in one or more preceding problems, a fact that instructors should bear in mind when assigning problems to a class.

While, as noted, this book is intended to serve as a textbook for a course, it is our hope that the wealth of carefully chosen examples and problems, together with the very explicit summary of prerequisite material in Part I, will enable it to be useful as well to the interested student who wishes to study functional analysis individually.

An instructor who plans to use this book as a textbook in a course has several options depending on the time available to him and the level of preparation of his students. He may wish to begin, for example, by devoting some weeks to the study of various chapters in Part I. Whether he does this or not, time limitations may make it impossible for him to treat all of Part II in one semester. With this in mind, we suggest the following abbreviated syllabus for a somewhat shorter course of study.

Chapter 11: Read entire text; omit Problems L–Q and U–Y.
Chapter 12: Read entire text; omit Problems R–Y.
Chapter 13: Read entire text; omit Problems S–T.
Chapter 14: Omit the material on Frechét spaces, viz., Examples H–L and Proposition 14.9; omit Problems Q–W.
Chapter 15: Omit all text after Theorem 15.11; omit Problems O–X.
Chapter 16: Omit the material on dual pairs, viz., everything after Proposition 16.12; omit Problems O–X.
Chapter 17: Read entire text; omit Problems T–Y.
Chapter 18: Omit the material on approximation theory, viz., everything after Example D; omit Problems V–W.
Chapter 19: Omit.

In the writing of this book no systematic effort has been made to attribute results or to assign historical priorities.

The notation and terminology used throughout the book are in essential agreement with those to be found in contemporary (American) textbooks. In particular, the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ will consistently represent the systems of positive integers, integers, real numbers, and complex numbers, respectively. We have also found it convenient to reserve the symbol $\mathbb{N}_0$ for the system of nonnegative integers.

Finally, there is one basic convention in force throughout the book: *All vector spaces that appear herein are either real or complex. If nothing is said about the scalar field of a vector space under discussion, it is automatically assumed to be complex.*

ARLEN BROWN
CARL PEARCY

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Contents

PART I
PRELIMINARIES

Set theory 1

We shall assume the reader to be familiar with the elements of set theory. Nonetheless, we begin with a review of certain set-theoretic fundamentals, largely to fix notation and terminology. (Readers wishing to improve their acquaintance with set theory, or to pursue in greater depth any of the topics touched on below, might consult [31] or [34]; another excellent source for most topics is [10].) For one thing, at the most elementary level, we reserve certain symbols throughout the book for several important sets. The system of positive integers is denoted by $\mathbb{N}$, the system of nonnegative integers by $\mathbb{N}_0$, the system of all integers by $\mathbb{Z}$, the real number system by $\mathbb{R}$, and the complex number system by $\mathbb{C}$. The empty set is denoted by $\varnothing$, and if X and Y are any two sets, the set-theoretic difference $\{x \in X : x \notin Y\}$ is denoted by $X \backslash Y$ and the symmetric difference $(X \backslash Y) \cup (Y \backslash X)$ by $X \nabla Y$. Moreover, if f is a mapping of X into Y (notation: $f: X \to Y$) and $A \subset X$ and $B \subset Y$, then $f(A)$ will denote the set $\{f(x) : x \in A\}$ and $f^{-1}(B)$ the set $\{x \in X : f(x) \in B\}$.

The reader is also assumed to be familiar with the notion of a partially ordered set. In this context our terminology and notation are quite standard. Thus if $X = (X, \leq)$ is a partially ordered set, then an element x_0 of X is *maximal* [*minimal*] in X if there exists no element x of X such that $x > x_0$ [$x < x_0$]. Likewise, if E is a subset of a partially ordered set X and if x_0 is an element of X such that $x \leq x_0$ for every x in E, then x_0 is an *upper bound* of E. If the set of upper bounds of E in X is nonempty, then E is *bounded above* in X. If, in addition, the set of upper bounds of E possesses a least element, then that *least upper bound* is also called the *supremum* of E and is denoted by $\sup E$. Dually E is *bounded below* if the set of lower bounds of E in X is nonempty; if, in addition, the set of lower bounds of E possesses a greatest element, then that *greatest lower bound* is called the *infimum* of E (notation: $\inf E$). A subset of a partially ordered set is *bounded* if it is bounded both above and below. For finite subsets $\{x_1, \ldots, x_n\}$ of a partially ordered set X we shall also write $x_1 \vee \cdots \vee x_n$ for $\sup\{x_1, \ldots, x_n\}$ and $x_1 \wedge \cdots \wedge x_n$ for $\inf\{x_1, \ldots, x_n\}$. If X has the property that $x \vee y$ and $x \wedge y$ exist for every pair of elements x and y of X, then X is a *lattice*. If, more generally, every subset of X has both a supremum and an infimum, then X is a *complete lattice*. A mapping f of one partially ordered set into another is *monotone increasing* [*decreasing*] if $x \le y$ implies $f(x) \le f(y)$ [$f(x) \ge f(y)$] and is *strictly* monotone increasing [decreasing] if $x < y$ implies $f(x) < f(y)$ [$f(x) > f(y)$]. A mapping f of a set X into a partially ordered set Y is *bounded* [*above, below*] if its range $f(X)$ is bounded [above, below] in Y.

Example A. The system $\mathbb{R}$ of real numbers is a lattice (in its usual ordering). Indeed, we have

$$s \vee t = \tfrac{1}{2}[s + t + |s - t|]$$

and

$$s \wedge t = \tfrac{1}{2}[s + t - |s - t|]$$

for every pair of real numbers s and t. If t is a real number the numbers $t \vee 0$ and $-(t \wedge 0)$ are called the *positive* and *negative parts* of t, and are denoted by t^+ and t^-, respectively. Note that t^+ and t^- are nonnegative and satisfy the conditions

$$t^+ + t^- = |t|,$$
$$t^+ - t^- = t,$$

for every real number t.

Example B. Every bounded nonempty subset of $\mathbb{R}$ has a supremum and an infimum in $\mathbb{R}$ (this is, in effect, one formulation of the *Dedekind postulate*; a lattice with this property is said to be *boundedly complete*). It follows that every closed interval $[a, b]$ $(=\{t \in \mathbb{R}: a \le t \le b\})$ is a complete lattice. While $\mathbb{R}$ itself is not a complete lattice, it is very useful to imbed $\mathbb{R}$ in a complete lattice. To do this we simply introduce two new "numbers," $+\infty$ and $-\infty$, and define $-\infty < +\infty$ and also $-\infty < t < +\infty$ for every t in $\mathbb{R}$. The enlarged set $\mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ is called the *extended real number system* and will consistently be denoted by $\mathbb{R}^\natural$. It is clear that $\mathbb{R}^\natural$ is a simply ordered complete lattice, and that if E is a subset of $\mathbb{R}$ that is not bounded above [below] in $\mathbb{R}$, then $\sup E = +\infty$ [$\inf E = -\infty$] in $\mathbb{R}^\natural$. We make a partial extension of the operation of addition to $\mathbb{R}^\natural$ by defining

$$t + (\pm\infty) = (\pm\infty) + t = \pm\infty$$

for every real number t, and

$$(\pm\infty) + (\pm\infty) = \pm\infty.$$