MATHEMATICAL METHODS OF PHYSICS

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PREFACE

For the last fourteen years a course in mathematical methods in physics has been taught, by the authors and by others, at the California Institute of Technology. It is intended primarily for first-year physics graduate students; in recent years most senior physics undergraduates, as well as graduate students from other departments, have been taking it. This book has evolved from the notes which have been used in this course.

We make little effort to teach physics in this text; the emphasis is on presenting mathematical techniques which have proved to be useful in analyzing problems in physics. We assume that the student has been exposed to the standard undergraduate physics curriculum: mechanics, electricity and magnetism, introductory quantum mechanics, etc., and that we are free to select examples from such areas.

In short, this is a book *about mathematics, for physicists.* Both motivation and standards are drawn from physics. That is, the choice of subjects is dictated by their usefulness in physics, and the level of rigor is intended to reflect current practice in theoretical physics.

It is assumed that the student has become acquainted with the following mathematical subjects:

1. Simultaneous linear equations and determinants
2. Vector analysis, including differential operations in curvilinear coordinates
3. Elementary differential equations
4. Complex variables, through Cauchy's theorem

However, these should not be considered as strict prerequisites. On the one hand, it will often profit the student to have had more background, and, on the other hand, it should not be too difficult for a student lacking familiarity with one of the above subjects to remedy the defect by extra work and/or outside reading. In fact, the subject of differential equations is discussed in the first chapter, partly to begin on familiar ground, and partly in order to treat some topics not normally covered in an elementay course in the subject.

A considerable variation generally exists in the amount of preparation different students have in the theory of functions of a complex variable. For this reason, we usually give a rapid review of the subject before studying contour integration (Chapter 3). Material for such a review is presented in the Appendix. Also, there are some excellent and reasonably brief mathematical books on the subject for the student unfamiliar with this material.

A considerable number of problems are included at the end of each chapter in this book. These form an important part of the course for which this book is designed. The emphasis throughout is on understanding by means of examples.

A few remarks may be made about some more or less unconventional aspects of the book. In the first place, the material which is presented does not necessarily flow in a smooth, logical pattern. Occasionally a new subject is introduced without the student's having been carefully prepared for the blow. This occurs, not necessarily through the irrationality or irascibility of the authors, but because that is the way physics is. Students in theoretical physics often need considerable persuasion before they will plunge into the middle of an unfamiliar subject; this course is intended to give practice and confidence in dealing with problems for which the student's preparation is incomplete.

A related point is that there is considerable deliberate nonuniformity in the depth of presentation. Some subjects are skimmed, while very detailed applications are worked out in other areas. If the course is to give practice in doing physics, the student must be given a chance to gain confidence in his ability to do detailed calculations. On the other hand, this is a text, not a reference work, and the material is intended to be fully covered in a year. It is therefore not possible to go into everything as deeply as one might like.

Several acknowledgments are in order. The course from which this text evolved was originally based on lectures by Professor R. P. Feynman at Cornell University. Much of Chapter 16 grew out of fruitful conversations with Dr. Sidney Coleman. The authors are grateful to Mrs. Julie Curcio for rapid, accurate, and remarkably neat typing through several revisions.

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CONTENTS

ONE

Ordinary Differential Equations

We begin this chapter with a brief review of some of the methods for obtaining solutions of an ordinary differential equation in closed form. Solutions in the form of power series are discussed in Section 1-2, and some methods for obtaining approximate solutions are treated in Sections 1-3 and 1-4.

The use of integral transforms in solving differential equations is discussed later, in Chapter 4. Applications of Green's function and eigenfunction methods are treated in Chapter 9, and numerical methods are described in Chapter 13.

1-1 SOLUTION IN CLOSED FORM

The *order* and *degree* of a differential equation refer to the derivative of highest order after the equation has been rationalized. Thus, the equation

$$\frac{d^3y}{dx^3} + x\sqrt{\frac{dy}{dx}} + x^2y = 0$$

is of third order and second degree, since when it is rationalized it contains the term $(d^3y/dx^3)^2$.

We first recall some methods which apply particularly to first-order equations. If the equation can be written in the form

$$A(x)\,dx + B(y)\,dy = 0 \tag{1-1}$$

we say the equation is *separable*; the solution is found immediately by integrating.

EXAMPLE

$$\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0 \tag{1-2}$$

$$\frac{dy}{\sqrt{1-y^2}} + \frac{dx}{\sqrt{1-x^2}} = 0$$

$$\sin^{-1} y + \sin^{-1} x = C$$

or, taking the sine of both sides,

$$x\sqrt{1-y^2} + y\sqrt{1-x^2} = \sin C = C'$$

More generally, it may be possible to integrate immediately an equation of the form

$$A(x, y)\,dx + B(x, y)\,dy = 0 \tag{1-3}$$

If the left side of (1-3) is the differential du of some function $u(x, y)$, then we can integrate and obtain the solution

$$u(x, y) = C$$

Such an equation is said to be *exact*. A necessary and sufficient condition that Eq. (1-3) be exact is

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x} \tag{1-4}$$

EXAMPLE

$$(x + y)\,dx + x\,dy = 0 \tag{1-5}$$

$$A = x + y \qquad B = x$$

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x} = 1$$

The solution is

$$xy + \tfrac{1}{2}x^2 = C$$

Sometimes we can find a function $\lambda(x, y)$, such that

$$\lambda(A\,dx + B\,dy)$$

is an exact differential, although $A\,dx + B\,dy$ may not have been. We call such a function λ an *integrating factor*. One can show that such factors always exist (for a first-order equation), but there is no general way of finding them.

Consider the general linear first-order equation

$$\frac{dy}{dx} + f(x)y = g(x) \tag{1-6}$$

Let us try to find an integrating factor $\lambda(x)$. That is,

$$\lambda(x)[dy + f(x)y\,dx] = \lambda(x)g(x)\,dx$$

is to be exact. The right side is all right, and our criterion (1-4) that the left side be exact is

$$\frac{d\lambda(x)}{dx} = \lambda(x)f(x)$$

This equation is separable, and its solution is

$$\lambda(x) = \exp\left[\int f(x)\,dx\right] \tag{1-7}$$

This is the integrating factor we were looking for.

EXAMPLE

$$xy' + (1 + x)y = e^x \tag{1-8}$$

$$y' + \left(\frac{1 + x}{x}\right)y = \frac{e^x}{x}$$

The integrating factor is $\exp\left\{\int [(1 + x)/x]\,dx\right\} = xe^x$

$$xe^x\left[y' + \left(\frac{1 + x}{x}\right)y\right] = e^{2x}$$

Now our equation is exact; integrating both sides gives

$$xe^xy = \int e^{2x}\,dx = \tfrac{1}{2}e^{2x} + C$$

$$y = \frac{e^x}{2x} + \frac{C}{x}e^{-x}$$

One can often simplify a differential equation by making a judicious change of variable. For example, the differential equation

$$y' = f(ax + by + c) \tag{1-9}$$

becomes separable if one introduces the new dependent variable,

$$v = ax + by + c$$

As another example, the so-called *Bernoulli equation*

$$y' + f(x)y = g(x)y^n \tag{1-10}$$

becomes linear if one sets $v = y^{1-n}$. (This substitution becomes "obvious" if the equation is first divided by y^n.)

A function $f(x, y, \ldots)$ in any number of variables is said to be *homogeneous* of degree r in these variables if

$$f(ax, ay, \ldots) = a^r f(x, y, \ldots)$$

A first-order differential equation

$$A(x, y)\, dx + B(x, y)\, dy = 0 \tag{1-11}$$

is said to be homogeneous if A and B are homogeneous functions of the same degree. The substitution $y = vx$ makes the homogeneous equation (1-11) separable.

EXAMPLE

$$y\, dx + (2\sqrt{xy} - x)\, dy = 0 \tag{1-12}$$

$$y = vx \qquad dy = v\, dx + x\, dv$$

$$vx\, dx + (2x\sqrt{v} - x)(v\, dx + x\, dv) = 0$$

$$2v^{3/2}\, dx + (2\sqrt{v} - 1)x\, dv = 0$$

This equation is clearly separable and its solution is trivial.

Note that this approach is related to dimensional arguments familiar from physics. A homogeneous function is simply a dimensionally consistent function, if $x, y, \ldots$ are all assigned the same dimension (for example, length). The variable $v = y/x$ is then a "dimensionless" variable.

This suggests a generalization of the idea of homogeneity. Suppose that the equation

$$A\, dx + B\, dy = 0$$

is dimensionally consistent when the dimensionality of y is some power m of the dimensionality of x. That is, suppose

$$\begin{aligned} A(ax, a^m y) &= a^r A(x, y) \\ B(ax, a^m y) &= a^{r-m+1} B(x, y) \end{aligned} \tag{1-13}$$

Such equations are said to be *isobaric*. The substitution $y = vx^m$ reduces the equation to a separable one.

EXAMPLE

$$xy^2(3y\,dx + x\,dy) - (2y\,dx - x\,dy) = 0 \tag{1-14}$$

Let us test to see if this is isobaric. Give x a "weight" 1 and y a weight m. The first term has weight $3m + 2$, and the second has weight $m + 1$. Therefore, the equation is isobaric with weight $m = -\frac{1}{2}$.

This suggests introducing the "dimensionless" variable $v = y\sqrt{x}$. To avoid fractional powers, we instead let

$$v = y^2x \qquad x = \frac{v}{y^2} \qquad dx = \frac{dv}{y^2} - \frac{2v\,dy}{y^3}$$

Equation (1-14) reduces to

$$(3v - 2)y\,dv + 5v(1 - v)\,dy = 0$$

which is separable.

An equation of the form

$$(ax + by + c)\,dx + (ex + fy + g)\,dy = 0 \tag{1-15}$$

where $a, \ldots, g$ are constants, may be made homogeneous by a substitution

$$x = X + \alpha \qquad y = Y + \beta$$

where α and β are suitably chosen constants [provided $af \neq be$; if $af = be$, Eq. (1-15) is even more trivial].

An equation of the form

$$y - xy' = f(y') \tag{1-16}$$

is known as a *Clairaut equation*. To solve it, differentiate both sides with respect to x. The result is

$$y''[f'(y') + x] = 0$$

We thus have two possibilities. If we set $y'' = 0$, $y = ax + b$, and substitution back into the original equation (1-16) gives $b = f(a)$. Thus $y = ax + f(a)$ is the general solution. However, we also have the possibility

$$f'(y') + x = 0$$

Eliminating y' between this equation and the original differential equation (1-16), we obtain a solution with no arbitrary constants. This is known as a *singular solution*.

EXAMPLE

$$y = xy' + (y')^2 \tag{1-17}$$

This is a Clairaut equation with general solution

$$y = cx + c^2$$

However, we must also consider the possibility $2y' + x = 0$. This gives

$$x^2 + 4y = 0$$

This singular solution is an *envelope* of the family of curves given by the general solution, as shown in Figure 1-1. The dotted parabola is the singular solution, and the straight lines tangent to the parabola are the general solution.

There are various other types of singular solutions, but we shall not go into them here. See COHEN for a good discussion and some references.

Next we review some methods which are useful for higher-order differential equations. An important type is the *linear equation with constant coefficients*:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x) \tag{1-18}$$

If $f(x) = 0$, the equation is *homogeneous*; otherwise it is *inhomogeneous*. Note that, if a linear equation is homogeneous, the sum of two solutions is also a solution, whereas this is not true if the equation is inhomogeneous.

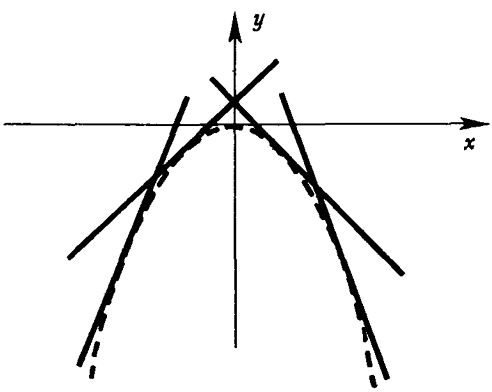

Figure 1-1 Solutions of the differential equation (1-17) and their envelope

The general solution of an inhomogeneous equation is the sum of the general solution of the corresponding homogeneous equation (the so-called *complementary function*) and any solution of the inhomogeneous equation (the so-called *particular integral*). This is in fact true for *any* linear differential equation, whether or not the coefficients are constants.

Solutions of the homogeneous equation [(1-18) with $f(x) = 0$] generally have the form

$$y = e^{mx}$$

Substitution into the homogeneous equation gives

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_0 = 0$$

If the n roots are $m_1, m_2, \ldots, m_n$ the complementary function is

$$c_1 e^{m_1 x} + \cdots + c_n e^{m_n x} \qquad (c_i \text{ are arbitrary constants})$$

Suppose two roots are the same, $m_1 = m_2$. Then we have only $n - 1$ solutions, and we need another. Imagine a limiting procedure in which m_2 approaches m_1. Then

$$\frac{e^{m_2 x} - e^{m_1 x}}{m_2 - m_1}$$

is a solution, and, as m_2 becomes m_1, this solution becomes

$$xe^{m_1 x}$$

This is our additional solution. If three roots are equal, $m_1 = m_2 = m_3$, then the three solutions are

$$e^{m_1 x} \qquad xe^{m_1 x} \qquad x^2 e^{m_1 x}$$

and so on.

A particular integral is generally harder to find. If $f(x)$ has only a finite number of linearly independent derivatives, that is, is a linear combination of terms of the form x^n, e^{ax}, $\sin kx$, $\cos kx$, or, more generally,

$$x^n e^{mx} \cos \alpha x \qquad x^n e^{mx} \sin \alpha x$$

then the method of *undetermined coefficients* is quite straightforward. Take for $y(x)$ a linear combination of $f(x)$ and its independent derivatives and determine the coefficients by requiring that $y(x)$ obey the differential equation.

EXAMPLE

$$y'' + 3y' + 2y = e^x \tag{1-19}$$

Complementary function:

$$m^2 + 3m + 2 = 0$$

$$m = -1, -2$$

$$y = c_1 e^{-x} + c_2 e^{-2x}$$

Particular integral: Try $y = Ae^x$. Substituting into the differential equation (1-19) gives

$$6A = 1 \qquad A = \tfrac{1}{6}$$

Thus, the general solution is

$$y = \tfrac{1}{6}e^x + c_1 e^{-x} + c_2 e^{-2x}$$

If $f(x)$, or a term in $f(x)$, is also part of the complementary function, the particular integral may contain this term and its derivatives multiplied by some power of x. To see how this works, solve the above example (1-19) with the right-hand side, e^x, replaced by e^{-x}.

There are several formal devices for obtaining particular integrals. If D means d/dx, then we can write our equation (1-18) as

$$(D - m_1)(D - m_2)\cdots(D - m_n)y = f(x) \tag{1-20}$$

A formal solution of (1-20) is

$$y = \frac{f(x)}{(D - m_1)\cdots(D - m_n)}$$

or, expanding by partial fraction techniques,

$$y = A_1 \frac{f(x)}{D - m_1} + \cdots + A_n \frac{f(x)}{D - m_n} \tag{1-21}$$

What does $f(x)/(D - m)$ mean? It is the solution of $(D - m)y = f(x)$, which is a first-order linear equation whose solution is trivial [see (1-6)].

Alternatively, we can just peel off the factors in (1-20) one at a time. That is,

$$(D - m_2)(D - m_3)\cdots(D - m_n)y = \frac{f(x)}{D - m_1} \tag{1-22}$$

We evaluate the right side, divide by $D - m_2$, evaluate again, and so on.

Finally, we consider the very important method known as *variation of parameters* for obtaining a particular integral. This method has the useful feature of applying equally well to linear equations with nonconstant coefficients. Before giving a general discussion of the method and applying it to an example, we shall digress briefly on the subject of *osculating parameters.*

Suppose we are given two linearly independent functions $y_1(x)$ and $y_2(x)$. By means of these we can define the two-parameter family of functions

$$c_1 y_1(x) + c_2 y_2(x) \tag{1-23}$$

Now consider some arbitrary function $y(x)$. Can we represent it by an appropriate choice of c_1 and c_2 in (1-23)? Clearly, the answer in general is no. Let us try the more modest approach of *approximating* $y(x)$ in the neighborhood of some fixed point $x = x_0$ by a curve of the family (1-23). Since there are two parameters at our disposal, a natural choice is to fit the *value* $y(x_0)$ and *slope* $y'(x_0)$ exactly. That is, c_1 and c_2 are determined from the two simultaneous equations

$$\begin{aligned} y(x_0) &= c_1 y_1(x_0) + c_2 y_2(x_0) \\ y'(x_0) &= c_1 y_1'(x_0) + c_2 y_2'(x_0) \end{aligned} \tag{1-24}$$

The c_1 and c_2 obtained in this way vary from point to point (that is, as x_0 varies) along the curve $y(x)$. They are called *osculating parameters* because the curve they determine fits the curve $y(x)$ as closely as possible at the point in question.

One can, of course, generalize to an arbitrary number N of functions y_i and parameters c_i. One chooses the c_i to reproduce the function $y(x)$ and its first $N - 1$ derivatives at the point x_0.

We now return to the problem at hand, solving linear differential equations. For simplicity, we shall restrict ourselves to second-order equations. Consider the inhomogeneous equation

$$p(x)y'' + q(x)y' + r(x)y = s(x) \tag{1-25}$$

and suppose we know the complementary function to be

$$c_1 y_1(x) + c_2 y_2(x)$$

Let us seek a solution of (1-25) of the form

$$y = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{1-26}$$

where the $u_i(x)$ are functions to be determined. In order to substitute (1-26) into (1-25), we must evaluate y' and y''. From (1-26),

$$y' = u_1 y_1' + u_2 y_2' + u_1' y_1 + u_2' y_2 \tag{1-27}$$

Before going on to y'', we observe that it would be convenient to impose the condition that the sum of the last two terms in (1-27) vanish, that is,

$$u_1' y_1 + u_2' y_2 = 0 \tag{1-28}$$