MICHAEL ATIYAH COLLECTED WORKS

VOLUME 1

Early Papers: General Papers

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Foreword to the Chinese Edition

When Michael Atiyah was interviewed by the Mathematical Intelligencer (vol. 6, pp. 9–19, 1984), he was asked about his most admired mathematician. He answered "Well, I think that is rather easy. The person I admire most is Hermann Weyl. He had interests in group theory, representation theory, differential equations, spectral properties of differential equations, differential geometry, theoretical physics; nearly everything I have done is very much in the spirit of the sort of things he worked in. And I entirely agree with his conceptions about mathematics and his view about what are the interesting things in mathematics." We find in these Collected Papers this mathematical philosophy and spirit preserved and continued.

I would like to advise my Chinese colleagues and students to take this as an advanced "textbook". No matter how refined or improved a new account is, the original papers on a subject are usually more direct and to the point. When I was young, I was benefited by the advice to read Henri Poircare, David Hilbert, Felix Klein, Adolf Hurwitz, etc. I did better with Wilhelm Blaschke, Elie Cartan, and Heinz Hopf This has also been in the Chinese tradition, when we were told to read Confucius, Han Yu in prose, and Tu Fu in poetry. It is my sincere hope that these Collected Papers will not be decorations on book shelves, but worn-out in the hands of young mathematicians.

陳省身

PREFACE

It appears to be increasingly fashionable to publish 'collected works' long before the author's demise. There are several clear advantages to all parties: posterity is saved the trouble of undertaking the collection, while the author can add some personal touches in the way of a commentary. There are also disadvantages: the commentary will be biased, and the author may feel that he is being pensioned off.

The initiative for these particular volumes came in fact from a different direction. A few years ago Professor Chern, who is now in active retirement trying to help China rebuild its mathematics, suggested that collections of mathematical papers made available in China would be most helpful to the younger Chinese mathematicians. Following on from this proposal the Oxford University Press agreed to publish my collected works and to make suitable arrangements to ensure their availability in China through the World Publishing Corporation in Beijing.

Essentially all my mathematical and quasi-mathematical publications are included here. The only exceptions are my textbook (with Ian Macdonald) on *Commutative algebra* and some articles which duplicate, identically or too closely, those published here. On the other hand I have included short articles, announcements of results or conference talks, which are later subsumed in larger papers. It seems to me that these still serve a useful purpose as a brief summary and introduction to the more technical papers.

There is always a problem deciding how to order papers in such a collection. The easiest course is to follow rigidly the date of publication, but this has little to commend it except inertia. The gap between submission and publication varies considerably and can run to two or three years. Also papers which have been published in several parts may not appear consecutively. Finally, any mathematical coherence can be lost in such a presentation with papers on different topics appearing all jumbled together. I have therefore tried to organize the material so that papers on related topics appear together, although the division is sometimes difficult and a bit arbitrary, for example in papers on the K-theory/Index theory boundary. Within each group I have broadly kept to a chronological order.

The commentaries I have provided are meant to fill in the mathematical background by explaining the genesis of ideas and their mutual relation. It is notorious that in mathematics the final published article, in attempting to clarify the logical presentation, usually obscures the origins and motivation. My commentaries are intended to rectify the situation in a small way. I have not hesitated to mention the names of colleagues and collaborators involved in the development of my ideas and, as far as possible, to describe their various contributions. I hope these personal touches will enhance the interest of the more formal material. Of course I realize that my memory may be faulty and, even worse, that by some subtle Freudian process I may have distorted the relative importance of what I have learnt from others. I apologize in advance to any who may have been unfairly treated.

I have indeed been fortunate to have had so many excellent mathematicians as my collaborators, and I thank all of them for allowing our joint papers to appear here. Above all I am indebted in many ways to my main collaborators, Raoul Bott, Fritz Hirzebruch, and Iz Singer. It has been a real pleasure to work with them over so many years.

Oxford
December 1986

M.F.A.

CURRICULUM VITAE

Born in London 22 April 1929, oldest son of Edward Atiyah and Jean Atiyah (née Levens).

Married 30 July 1955 to Lily Brown. Three sons, John, David, Robin.

Knight Bachelor 1983

Education:

(Primary) Diocesan School, Khartoum, Sudan 1934–41.

(Secondary) Victoria College, Cairo & Alexandria, Egypt, 1941–45. Manchester Grammar School, 1945–47.

National Service R.E.M.E. 1947–49.

Trinity College, Cambridge, B.A., 1952, Ph.D. 1955. Research Fellow, 1954–58.

Commonwealth Fund Fellow, The Institute for Advanced Study, Princeton, 1955–56.

Tutorial Fellow, Pembroke College, Cambridge, 1958–61.

Assistant Lecturer, Cambridge University, 1957–58, Lecturer, 1958–61.

Reader, Oxford University and Professorial Fellow of St. Catherine's College, 1961–63.

Savilian Professor of Geometry, Oxford University and Professorial Fellow, New College, 1963–69.

Professor of Mathematics, The Institute for Advanced Study, Princeton, 1969–72.

Royal Society Research Professor, Oxford University and Professorial Fellow of St. Catherine's College, 1973–

Fellow of the Royal Society and Foreign member of: National Academy of Sciences USA, American Academy of Arts and Sciences, Academie des Sciences (France), Akademie Leopoldina, Royal Swedish Academy, Royal Irish Academy, Royal Society of Edinburgh.

Doctor honoris causa of Universities of Bonn, Warwick, Durham, St. Andrews, Dublin, Chicago, Cambridge, Edinburgh, Essex, London, Sussex, Ghent.

President, London Mathematical Society 1975–77, Mathematical Association 1981–82, Vice-President Royal Society 1984–85.

Fields Medal, Moscow, 1966.

EARLY PAPERS 1–9
(1952–58)

My first paper [1] was written when I was a second-year undergraduate and arose from a course on higher-dimensional projective geometry given by J. A. Todd. At that time I was fascinated by classical projective geometry, a subject well represented at the time in Cambridge by Todd, Babbage, and White with the ghost of H. F. Baker still prominent in the background. Todd was my supervisor for several terms and he arranged for my little note to be published, something which gave me disproportionate pleasure and encouragement. When I came to decide on my graduate work I oscillated between Todd and Professor W. V .D. Hodge who represented a more modern approach based on differential geometry. Hodge's greater international standing swung the balance and, in 1952, I became his research student.

In my first year Newton Hawley from America was visiting Cambridge and through him I became interested in analytic fibre bundles. At the same time great things were happening in France and I was an avid reader of the *Comptes Rendus*, following the developments in sheaf theory. Peter Hilton also showed me a letter of Serre (addressed to André Weil), that was circulating at the time, which gave the sheaf-theory treatment of Riemann–Roch for an algebraic curve. Given my interest in classical geometry I naturally looked at the old results on ruled surfaces from the new point of view and this led to [2], for which I received the Smith's Prize in 1954. This came at a crucial time when I was unsure whether I should continue with mathematical research. In fact I toyed quite seriously with subjects like architecture and archeology, but the Smith's Prize decided my fate.

The role of fibre bundles in algebraic geometry had been hinted at in André Weil's 1938 paper and was developed by him in his Chicago lectures in the early fifties. He spent a term in Cambridge in 1953 lecturing on the topic but I received no encouragement from this quarter. In subsequent years this whole subject has seen extensive development in different directions. The particular problems studied in [2] concerning the classification of fibre bundles over curves were taken up more systematically by the Tata Institute School (Narasimhan, Seshadri, Ramanan) and also by my student Newstead.

My supervisor Hodge took a keen interest in my work and he also was following the new developments in algebraic geometry. One day he outlined to me the way in which Lefschetz's theorems on integrals of the second kind should fit into the sheaf-theory framework. I developed this idea in great detail, resulting eventually in our joint paper [4] (summarized in [3]) which Hodge reported on when he attended a major conference in Princeton. This attracted the attention of Kodaira and Spencer and was instrumental in my going to the Institute for Advanced Study in 1955.

Commentary

Princeton in 1955–56 was enormously stimulating. In particular, I got to know Serre, Hirzebruch, Bott, and Singer all at this time, with long-term implications for my subsequent work. Serre ran a seminar on vector bundles which I attended and my next three papers [5], [6], [7] were all influenced by him in one way or another.

Back in England I met Milnor, then in Oxford, and had a discussion with him about Kummer surfaces. Attempting to understand the effect of double points on the topology of algebraic surfaces then led to [8]. This paper was later the starting point for Brieskorn's beautiful work on rational double points.

Around this time (1957) the first of Hirzebruch's *Arbeitstagungs* began my long series of visits to Bonn. [9] was an expository lecture given there based on work of Calabi which I had learnt about in Princeton.

A NOTE ON THE TANGENTS OF A TWISTED CUBIC

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A NOTE ON THE TANGENTS OF A TWISTED CUBIC

By M. F. ATIYAH

Communicated by J. A. Todd

Received 8 May 1951

1. Consider a rational normal cubic C_3. In the Klein representation of the lines of S_3 by points of a quadric Ω in S_5, the tangents of C_3 are represented by the points of a rational normal quartic C_4. It is the object of this note to examine some of the consequences of this correspondence, in terms of the geometry associated with the two curves.

2. C_4 lies on a Veronese surface V, which represents the congruence of chords of C_3 (1). Also C_4 determines a 4-space Σ meeting Ω in Ω_1, say; and since the surface of tangents of C_3 is a developable, consecutive tangents intersect, and therefore the tangents to C_4 lie on Ω, and so on Ω_1. Hence Ω_1, containing the sextic surface of tangents to C_4, must be the quadric threefold I associated with C_4, i.e. the quadric determining the same polarity as C_4 (2). We note also that the tangents to C_4 correspond in S_3 to the plane pencils with vertices on C_3, and lying in the corresponding osculating planes.

3. We shall prove that the surface U, which is the locus of points of intersection of pairs of osculating planes of C_4, is the projection of the Veronese surface V from L, the pole of Σ, on to Σ.

Let P denote a point of C_3, and t, π the tangent line and osculating plane at P, and let T, τ, ϖ denote the same for the corresponding point of C_4. Further, let $\varpi_1 \varpi_2$ meet in Q, which is therefore a point of U, and let LQ meet Ω in R, R'. We show that R and R' represent the two lines P_1P_2 and (π_1, π_2), and therefore that R, or R', is a point of V.

The polar 3-space of Q with respect to I, being the same as that with respect to C_4, meets C_4 twice at each of T_1 and T_2, and therefore contains τ_1 and τ_2. But this 3-space is the polar 3-space of LQ with respect to Ω. Hence R and R', being conjugate to all the points of τ_1 and τ_2, represent lines in S_3 which intersect all lines of the pencils (P_1, π_1) (P_2, π_2), and so must be the lines P_1P_2, (π_1, π_2) as stated.

4. We now give a geometrical proof of the well-known result: a necessary and sufficient condition for four points on C_4 to be equianharmonic is that the pole of the 3-space determined by them should lie on I (i.e. that the 3-space should touch I).

Let A, B, C be three points on C_4, and let T be a collineation on C_4, such that

$$T(ABC) = (BCA).$$

Then $T^3 = 1$, the identical collineation, and $T(H_1H_2) = (H_1H_2)$, where H_1, H_2 are the Hessian pair of A, B, C. Since C_4 is a rational normal curve, there is a unique collineation S of S_4 which induces T on C_4, and $S^3 = 1$.

Now, under the collineation S, the plane ABC and the quadric I remain fixed. Hence the two tangent primes from ABC to I remain fixed (they cannot interchange since $S^3 = 1$). Hence their fourth points of intersection with C_4, being fixed, must be H_1 and H_2, i.e. the primes $ABCH_1$, $ABCH_2$ touch I. But, in any equianharmonic range, every point is one of the Hessian pair of the other three. This proves the necessity of the condition. Conversely, if $ABCD$ touches I, then D must be H_1 or H_2, and $(ABCD)$ is equianharmonic.

5. Interpreting this last result for S_3, we see that it is equivalent to: the tangents at four points of C_3 have a unique transversal if, and only if, the four points are equianharmonic.

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(1) BAKER, H. F. *Principles of geometry* (Cambridge, 1940), vol. 4, p. 52.
(2) TELLING, H. G. *The rational quartic curve in space of three and four dimensions* (Camb. Tracts Math. no. 34, 1936), p. 8.

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COMPLEX FIBRE BUNDLES AND RULED SURFACES

COMPLEX FIBRE BUNDLES AND RULED SURFACES

By M. F. ATIYAH

[Received 5 August 1954.—Read 25 November 1954]

Introduction

ALTHOUGH much work has been done in the topological theory of fibre bundles, very little appears to be known on the complex analytic side. In this paper we propose to study certain types of complex fibre bundle, showing how they can be classified. The methods we shall employ will be based on the theory of stacks, for the full details of which we refer the reader to (**2**). We shall, however, recall all the basic definitions and results in so far as they are necessary for the applications we have in view.

In section I, after making the initial definitions, we derive the classification theorem for one-dimensional affine bundles over an algebraic variety (Theorem 1), and we prove that all such bundles are 'regular' (Theorem 2). In section II we restrict attention to the case in which the base space is an algebraic curve. The fibre bundles we consider are then essentially algebraic ruled surfaces, and our aim is to provide an 'intrinsic' classification for them. This is simply the classification problem for one-dimensional projective bundles, and we show how it can be reduced to the study of a curve in projective space (Theorem 5). For low values of the genus we obtain explicit solutions of this problem (Theorems 6.0, 6.1, 6.2). Finally in § 5 we give a brief discussion of tangent bundles. Some points raised by the work of Hawley are then examined in an Appendix.

I. FIBRE BUNDLES OVER ALGEBRAIC VARIETIES

1. Cohomology theory of stacks

Let X be a compact complex manifold (which we shall later restrict to be algebraic), and let S be a given category of functions locally defined on X; here we use the word 'category' in a general sense, *not* in the technical sense of Eilenberg and Steenrod (**5**). For example S might be the category of locally holomorphic functions, locally meromorphic functions, or locally holomorphic differential forms. Then cohomology groups $H^q(X, S)$ can be constructed as follows. Let α be a finite covering of X by open sets $\{U_i\}$, N_α its nerve (see (**5**)). If $\sigma^q = \sigma^q_{i_0 i_1 \ldots i_q}$ is a q-simplex of N_α, we denote by $|\sigma^q|$ its *support* in X, that is the open set $U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_q}$. We denote by $C^q(\alpha, S)$ the q-cochain group with coefficients in S, a cochain c^q being

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a sum $\sum_j s_j \sigma_j^q$ over all q-simplexes σ_j^q of N_α, with s_j a function of the category S *defined throughout* $|\sigma_j^q|$. $C^q(\alpha, S)$ has a coboundary operator δ induced from the coboundary operator of the complex N_α; in detail, we define

$$\delta\Big(\sum_j s_j \sigma_j^q\Big) = \sum_{j,k} \bar{s}_j \eta_{jk} \sigma_k^{q+1},$$

where $\delta\sigma_j^q = \sum_k \eta_{jk} \sigma_k^{q+1}$ is the coboundary of σ_j^q in N_α, and $\bar{s}_j$ is the restriction of s_j from $|\sigma_j^q|$ to $|\sigma_k^{q+1}|$ (this latter being a subset of the former when σ_j^q is a face of σ_k^{q+1}). If we denote by $Z^q(\alpha, S)$, $B^q(\alpha, S)$ the cocycles and coboundaries respectively, we define the cohomology groups by

$$H^q(\alpha, S) = Z^q(\alpha, S)/B^q(\alpha, S).$$

If now β is a refinement of α, there exists a simplicial map $\tau_{\beta\alpha}$: $N_\beta \to N_\alpha$. As in the ordinary Čech theory, any such $\tau_{\beta\alpha}$ induces a homomorphism $\tau_{\alpha\beta}^*$: $H^q(\alpha, S) \to H^q(\beta, S)$ which is independent of the particular $\tau_{\beta\alpha}$ chosen. Furthermore $\tau_{\alpha\beta}^* \tau_{\beta\gamma}^* = \tau_{\alpha\gamma}^*$, so that we can define the direct limit of $H^q(\alpha, S)$ with respect to the directed set formed by the coverings $\{\alpha\}$ (see (**5**)). This direct limit is $H^q(X, S)$, the qth cohomology group of X with coefficients in S. In all our applications the functions of S will form a complex vector space, and the cohomology groups will therefore be vector spaces also.

The category S defines a *stack* (faisceau) (see (**2**)) over X which we shall also denote by S. Besides possessing a number of obvious properties, to which no explicit reference will be made, the stack cohomology groups also satisfy an exact sequence theorem, which states that if

$$0 \to S' \to S \to S'' \to 0 \tag{1}$$

is an exact sequence of stacks on X, then the corresponding cohomology sequence

$$\to H^{q-1}(X, S'') \to H^q(X, S') \to H^q(X, S) \to H^q(X, S'') \to H^{q+1}(X, S') \to$$

is also exact. We recall that a sequence of homomorphisms is said to be exact if the image of each homomorphism is the kernel of the next; in particular, an exact sequence of the form (1) simply asserts that S'' is the quotient stack S/S', which in our case means that S'' is the category of functions of S modulo functions of S'.

Another important property, which this time follows immediately from the definition, is that $H^0(X, S)$ is just the group of functions of S defined on the *whole* of X.

2. Complex analytic bundles

An analytic fibre bundle is a fibre bundle in which everything has an analytic structure; more precisely, the base space, the fibre, and the bundle space are all complex analytic manifolds, the group is a group of analytic