LOGIC FOR MATHEMATICIANS

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Preface

Every mathematician must know the conversation-stopping nature of the reply he gives to an inquiry by a non-mathematician about the nature of his business. For a logician in the company of other mathematicians to admit his calling is to invite similarly blank looks, admissions of ignorance, and a change in the topic of conversation. The rift between mathematicians and the public is a difficulty which will always exist (though no opportunity should be missed of narrowing it), but the rift between logicians and other mathematicians is, in my view, unnecessary. This book is an attempt to bridge the gap by providing an introduction to logic for mathematicians who do not necessarily aspire to becoming logicians.

Mathematical logic is now taught in many universities as part of an undergraduate course in mathematics or computing, and the subject is now coherent enough to have a standard body of fundamental material which must be included in any first course. This book is intended to be a textbook for such a course, but also to be something more – to be a book rather than merely a textbook. The material is deliberately presented in a direct manner, for its own sake, without particular bias towards any aspect, application or development of the subject. At the same time the attempt has been made to place the subject matter in the context of mathematics as a whole and to emphasise the relevance of logic to the mathematician.

The book is designed to be accessible to anyone with a mathematical background, from first year undergraduate to professional mathematician, who wishes, or is required, to find out something of what mathematical logic is. A certain familiarity with elementary algebra and number theory is assumed, and since ideas of countable and uncountable sets are fundamental, there is an appendix in which the necessary properties are described.

The material of the book is developed from that presented in two separate courses of sixteen lectures at the University of Stirling to students in their third and fourth years of undergraduate study. The first of these covered Chapters 1 to 4 with some of Chapter 5, and the second was a more advanced optional course covering the remainder. Chapter 6 is the most difficult in the book, but the significance of Gödel's Incompleteness Theorem is such that the ideas behind the proof ought to be brought out in a book of this kind. The detailed proofs may be omitted on first reading, as the material of Chapter 7 does not depend on them.

The scope of this book is more limited than that of other standard introductions to the subject. In particular the theory of models and the axiomatic theory of sets are barely touched on. The interested reader is therefore referred to the list at the end of the book of titles for further reading. Some of these are specifically referred to in the text (by the author's name), and overall they provide coverage of most areas of mathematical logic and treat in more depth the topics of this book.

There are exercises at the end of each section. Generally speaking, routine examples precede more taxing ones, but all the examples are intended as direct applications of the material in the corresponding section. Their purpose is to clarify and consolidate that material, not to extend it. Hints or solutions to many of the exercises are provided at the end of the book.

The symbols used in the book are, as far as is possible, standard (as is the terminology). There are some non-standard usages, however, which have been introduced in order to achieve clarity. These ought not to trouble the reader who is familiar with the material, and are intended to help the reader who is not. It is an unfortunate fact that different authors do use different notations and symbolism. For this reason, and for ease of reference, a glossary of symbols is included. Throughout the text the symbol $\triangleright$ is used to denote the resumption of the main exposition after it has been broken by a proposition, example, remark, corollary or definition.

Finally there are four debts which I wish to acknowledge. First, my debt to the book by Mendelson (*Introduction to Mathematical Logic*) will be apparent to all who are familiar with it. As a basic text for logicians it has had few rivals. Second, this book would not have been possible but for the time made available to me by the University of Stirling. Third, on a more personal level, I am most grateful to Francis Bell for his conscientious reading of a draft of the text and for his numerous valuable suggestions. And last, my sincere thanks go to Irene Wilson and May Abrahamson for all their patient labour in typing the manuscript.

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Contents

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Informal statement calculus

1.1 Statements and connectives

Logic, or at least logical mathematics, consists of deduction. We shall examine the rules of deduction making use of the precision which characterises a mathematical approach. In doing this, if we are to have any precision at all we must make our language unambiguous, and the standard mathematical way of doing that is to introduce a symbolic language, with the symbols having precisely stated meanings and uses. First of all we shall examine an aspect of everyday language, namely *connectives* (or conjunctions†, which is the more common grammatical term).

When we try to analyse a sentence in the English language, we can first note whether it is a simple sentence or a compound sentence. A simple sentence has a subject and a predicate (in the grammatical sense), for example:

> Napoleon is dead
> John owes James two pounds
> All eggs which are not square are round.

In each case the subject is underlined, and the remainder is the predicate. A compound sentence is made up from simple sentences by means of connectives, for example:

> Napoleon is dead and the world is rejoicing
> If all eggs are not square then all eggs are round
> If the barometer falls then either it will rain or it will snow.

We shall regard it as a basic assumption that all the simple sentences which we consider will be either true or false. It could certainly be argued that there are sentences which could not be regarded as either true or false, so we shall use a different term. We shall refer to simple and compound *statements*, and our assumption will be that all statements are either true or false.

† The word 'conjunction' has a more specific meaning for us. It is defined in Section 1.2.

Simple statements will be denoted by capital letters A, B, C, ... so in order to symbolise compound statements we have to introduce symbols for the connectives. The most common connectives, and the symbols which we shall use to denote them, are given in the table below.

not A	$\sim A$
A and B	$A \wedge B$
A or B	$A \vee B$
if A then B	$A \rightarrow B$
A if and only if B	$A \leftrightarrow B$

Of course, if the meaning of the symbols is to be defined precisely, we must be sure that we know precisely the meanings of the expressions in the left hand column. We shall return to this shortly.

The three compound statements above could be written in symbols (respectively) thus:

$$A \wedge B$$

$$C \rightarrow D$$

$$E \rightarrow (F \vee G)$$

where A stands for 'Napoleon is dead', B stands for 'the world is rejoicing', C stands for 'all eggs are not square', etc.

Notice that what remains when a compound statement is symbolised in this way is the bare logical bones, a mere 'statement form', which several different statements might have in common. It is precisely this which enables us to analyse deduction. For deduction has to do with the 'forms' of the statements in an argument rather than their meanings.

Example 1.1

If Socrates is a man then Socrates is mortal
Socrates is a man
∴ Socrates is mortal.

This is an argument which is regarded as logically satisfactory. But consider the argument:

Socrates is a man
∴ Socrates is mortal.

The conclusion may be thought to follow from the premiss, but it does so because of the meanings of the words 'man' and 'mortal', not by a mere logical deduction. Let us put these arguments into symbols.

$A \rightarrow B$	A
A	$\therefore B$
$\therefore B$	

It is the 'form' of the first which makes it valid. Any argument with the same form would also be valid. This is our logical intuition about if . . . then . . . statements. However, the second does not share this property. There are many arguments of this form which we would not regard intuitively as valid, For example:

The moon is yellow
∴ The moon is made of cheese.

We study, therefore, *statement forms* rather than particular statements. The letters $p, q, r, \ldots$ will be *statement variables* which stand for arbitrary and unspecified simple statements. Notice the distinction between the usages of the letters $p, q, r, \ldots$ and the letters $A, B, C, \ldots$ The former are variables for which particular simple statements may be substituted. The latter are merely 'labels' for particular simple statements. The variables enable us to describe in general the properties that statements and connectives have. Now each simple statement is either true or false, so a given statement variable can be thought of as taking one or other of the two *truth values*: T (true) or F (false). The way in which the truth or falsity of a compound statement or statement form depends on the truth or falsity of the simple statements or statement variables which constitute it is the subject of the next section.

Exercises

1 Translate into symbols the following compound statements.
- (*a*) If demand has remained constant and prices have been increased, then turnover must have decreased.
- (*b*) We shall win the election, provided that Jones is elected leader of the party.
- (*c*) If Jones is not elected leader of the party, then either Smith or Robinson will leave the cabinet, and we shall lose the election.
- (*d*) If x is a rational number and y is an integer, then z is not real.
- (*e*) Either the murderer has left the country or somebody is harbouring him.
- (*f*) If the murderer has not left the country, then somebody is harbouring him.
- (*g*) The sum of two numbers is even if and only if either both numbers are even or both numbers are odd.
- (*h*) If y is an integer then z is not real, provided that x is a rational number.

2 (*a*) Pick out any pairs of statements from the list in Exercise 1 which have the same form.
- (*b*) Pick out any pairs of statements from the list in Exercise 1 which have the same meaning.

1.2 Truth functions and truth tables

Let us consider the connectives in turn.

Negation

The negation of a statement A we write $\sim A$. Clearly if A is true then $\sim A$ is false, and if A is false then $\sim A$ is true. The meaning of A is irrelevant. We can describe the situation by a *truth table*:

p	$\sim p$
T	F
F	T

The table gives the truth value of $\sim p$, given the truth value of p. The connective $\sim$ gives rise to a *truth function*, $f^{\sim}$, in this case a function from the set $\{T, F\}$ to itself, given by the truth table, thus:

$$f^{\sim}(T) = F,$$

$$f^{\sim}(F) = T.$$

Conjunction

As above, it is easy to see that the truth value taken by the conjunction $A \wedge B$ of two statements A and B depends only on the truth value taken by A and the truth value taken by B. We have the table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

We have in the table one row for each of the possible combinations of truth values for p and q. The last column gives the corresponding truth values for $p \wedge q$. The connective $\wedge$ thus defines a truth function $f^{\wedge}$ of two places.

$$f^{\wedge}(T, T) = T,$$

$$f^{\wedge}(T, F) = F,$$

$$f^{\wedge}(F, T) = F,$$

$$f^{\wedge}(F, F) = F.$$

Disjunction

We have used $A \vee B$ to denote 'A or B,' but there are two distinct standard usages of the word 'or' in English. 'A or B' may mean 'A or B or both', or it may mean 'A or B but not both'. In order to keep our symbolic language precise we must choose only one of these to give the meaning of our symbol $\vee$. We choose the former. There is no particular reason for this; we could just as well have chosen the latter. The truth table is as follows:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The connective $\vee$ defines a truth function of two places just as $\wedge$ did.

Remark. If A and B are simple statements, we can symbolise 'A or B but not both' as

$$(A \vee B) \wedge \sim(A \wedge B).$$

Correspondingly, if we had used 'A or B but not both' to define our disjunction symbol, we could have expressed 'A or B or both' using that disjunction along with $\wedge$ and $\sim$.

Conditional

$A \rightarrow B$ is to represent the statement 'A implies B' or 'if A then B'. Now in this case normal English usage is not as helpful in constructing a truth table, and the table that we use is a common source of intuitive difficulty. It is:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The difficulty arises with the truth value T assigned to $A \rightarrow B$ in the cases where A is false. Consideration of examples of conditional statements in which the antecedent is false might perhaps lead one to the conclusion that such statements do not have a truth value at all. One might also gain the impression that such statements are not useful or meaningful. For example, the statement:

If grass is red then the moon is made of green cheese

could fairly be said to be meaningless.

However, we shall be interested in deduction and methods of proof, principally in mathematics. In this context the significance of a conditional statement $A \rightarrow B$ is that its truth enables the truth of B to be inferred from the truth of A, and nothing in particular to be inferred from the falsity of A. A very common sort of mathematical statement can serve to illustrate this, namely a *universal* statement, for example:

For every integer n, if $n > 2$ then $n^2 > 4$.

This is regarded as a true statement about integers. We would expect, therefore, to regard the statement

If $n > 2$ then $n^2 > 4$

as true, irrespective of the value taken by n. Different values of n give rise to all possible combinations of truth values for '$n > 2$' and '$n^2 > 4$' except the combination *TF*. Taking n to be 3, -1, 1 respectively yields the combinations *TT*, *FT*, *FF*, and these are the combinations which, according to our truth table, give the implication the truth value *T*. The intuitive truth of this implication is therefore some justification for the truth table. The point to remember is that the only circumstance in which the statement $A \rightarrow B$ is regarded as false is when A is true and B is false.

Biconditional

We denote 'A if and only if B' by $A \rightarrow B$. The situation here is clear. We should have $A \leftrightarrow B$ true when and only when A and B have the same truth value (both true or both false). The truth table is then as shown.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

This completes our list of connectives. Obviously, compound statements of any length can be built up from simple statements using these connectives. Using statement variables we can build up statement *forms* of any length.

Definition 1.2

A *statement form* is an expression involving statement variables and connectives, which can be formed using the rules:

(i) Any statement variable is a statement form.

(ii) If $\mathscr{A}$ and $\mathscr{B}$ are statement forms, then $(\sim\mathscr{A})$, $(\mathscr{A} \wedge \mathscr{B})$, $(\mathscr{A} \vee \mathscr{B})$, $(\mathscr{A} \rightarrow \mathscr{B})$, and $(\mathscr{A} \leftrightarrow \mathscr{B})$ are statement forms.

Example 1.3

$((p \wedge q) \rightarrow (\sim(q \vee r)))$ is a statement form. By (i), p, q, r are statement forms. By (ii), $(p \wedge q)$ and $(q \vee r)$ are statement forms. By (ii), $(\sim(q \vee r))$ is a statement form. By (ii), $((p \wedge q) \rightarrow (\leftarrow(q \vee r)))$ is a statement form.

▷ This definition is an example of an inductive definition. It sets a pattern which will occur again when we describe formal systems in detail.

The connectives determine simple truth functions. Using the truth tables for the connectives, we can construct a truth table for any given statement form. By this is meant a table which will indicate, for any given assignment of truth values to the statement variables appearing in the statement form, the truth value which it takes. This truth table is a graphical representation of a truth function. Thus each statement form gives rise to a truth function, the number of arguments of the function being the number of different statement variables appearing in the statement form. Let us illustrate this by means of some examples.

Example 1.4

(*a*) $((\sim p) \vee q)$.

First construct the truth table:

p	q	$(\sim p)$	$((\sim p) \vee q)$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Observe that the truth function corresponding to this statement form is the same as the truth function determined by $(p \rightarrow q)$.

(*b*) $(p \rightarrow (q \vee r))$.

Truth table:

p	q	r	$(q \vee r)$	$(p \to (q \vee r))$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	F	T

The truth function here is a three place function, since there are three statement variables. Each row of the table gives the value of the truth function for a different combination of truth values for the letters. Notice that there will be eight rows in the truth table of any statement form involving three statement variables, and notice the pattern in which the first three columns of the table above are written out. This way of grouping the Ts and Fs under the p, q, r ensures that each possible combination appears once and only once.

▷ In the general case, for a statement form involving n different statement variables (n any natural number), the truth function will be a function of n places, and the truth table will have 2^n rows, one for each of the possible combinations of truth values for the statement variables. Further, notice that there are 2^{2^n} distinct truth functions with n places, corresponding to the 2^{2^n} possible ways of arranging the Ts and Fs in the last column of a truth table with 2^n rows. The number of statement forms which can be constructed using n statement variables is clearly infinite, so it follows that different statement forms may correspond to the same truth function.

To investigate this further we need some definitions.

Definition 1.5

(a) A statement form is a *tautology* if it takes truth value T under each possible assignment of truth values to the statement variables which occur in it.

(b) A statement form is a *contradiction* if it takes truth value F under each possible assignment of truth values to the statement variables which occur in it.

▷ Not every statement form falls into one or other of these categories. In fact none of those which we have considered so far does.

Example 1.6

- (*a*) $(p \vee \sim p)$ is a tautology.
- (*b*) $(p \wedge \sim p)$ is a contradiction.
- (*c*) $(p \leftrightarrow (\sim(\sim p)))$ is a tautology.
- (*d*) $(((\sim p) \rightarrow q) \rightarrow (((\sim p) \rightarrow (\sim q)) \rightarrow p))$ is a tautology.

The method used to verify that a given statement form is either a tautology or a contradiction is just construction of the truth table.

▷ It should be clear from the definition that all tautologies containing n statement variables give rise to the same truth function of n places, namely that which takes value T always. We can make a similar observation about contradictions.

Definition 1.7

If $\mathscr{A}$ and $\mathscr{B}$ are statement forms, $\mathscr{A}$ *logically implies* $\mathscr{B}$ if $(\mathscr{A} \rightarrow \mathscr{B})$ is a tautology, and $\mathscr{A}$ is *logically equivalent* to $\mathscr{B}$ if $(\mathscr{A} \leftrightarrow \mathscr{B})$ is a tautology.

Example 1.8

- (*a*) $(p \wedge q)$ logically implies p.
- (*b*) $(\sim(p \wedge q))$ is logically equivalent to $((\sim p) \vee (\sim q))$.
- (*c*) $(\sim(p \vee q))$ is logically equivalent to $((\sim p) \wedge (\sim q))$.

For (*a*): truth table of $((p \wedge q) \rightarrow p)$:

$((p$	$\wedge$	$q)$	$\rightarrow$	$p)$
T	T	T	T	T
T	F	F	T	T
F	F	T	T	F
F	F	F	T	F

For (*b*):

$(\sim$	$(p$	$\wedge$	$q)$	$\leftrightarrow$	$((\sim$	$p)$	$\vee$	$(\sim$	$q)))$
F	T	T	T	T	F	T	F	F	T
T	T	F	F	T	F	T	T	T	F
T	F	F	T	T	T	F	T	F	T
T	F	F	F	T	T	F	T	T	F

Here we have introduced a different way of writing the truth tables. For complicated statement forms it is easier to construct the table this way. Start by writing columns of Ts and Fs under the statement variables in the same order as we have in previous tables, to ensure that each combination appears once only. This must be done consistently throughout, of course. Next, under the connectives successively insert the truth values of the parts, until the column giving the truth values of the whole statement form is filled up. This column is enclosed by vertical lines in the examples above.

Remark. Let $\mathscr{A}$ and $\mathscr{B}$ be statement forms containing the same statement variables. If $\mathscr{A}$ and $\mathscr{B}$ are logically equivalent then they represent the same truth function. For if $(\mathscr{A} \leftrightarrow \mathscr{B})$ is a tautology it never takes value F, and so $\mathscr{A}$ and $\mathscr{B}$ must always take the same truth value. The truth functions corresponding to $\mathscr{A}$ and $\mathscr{B}$ must therefore be the same.

Exercises

3 Write out the truth tables of the following statement forms:

(*a*) $((\sim p) \wedge (\sim q))$;
(*b*) $\sim((p \rightarrow q) \rightarrow (\sim(q \rightarrow p)))$;
(*c*) $(p \rightarrow (q \rightarrow r))$;
(*d*) $((p \wedge q) \rightarrow r)$;
(*e*) $((p \leftrightarrow (\sim q)) \vee q)$;
(*f*) $((p \wedge q) \vee (r \wedge s))$;
(*g*) $(((\sim p) \wedge q) \rightarrow ((\sim q) \wedge r))$;
(*h*) $((p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r)))$.

4 Show that the statement form $((\sim p) \vee q)$ gives rise to the same truth function as $(p \rightarrow q)$, and that $((\sim p) \rightarrow (q \vee r))$ gives rise to the same truth function as $((\sim q) \rightarrow ((\sim r) \rightarrow p))$.

5 Which of the following statement forms are tautologies?

(*a*) $(p \rightarrow (q \rightarrow p))$;
(*b*) $((q \vee r) \rightarrow ((\sim r) \rightarrow q))$;
(*c*) $((p \wedge (\sim q)) \vee ((q \wedge (\sim r)) \vee (r \wedge (\sim p))))$;
(*d*) $((p \rightarrow (q \rightarrow r)) \rightarrow ((p \wedge (\sim q)) \vee r))$.

6 Show that the following pairs of statement forms are logically equivalent.

(*a*) $(p \rightarrow q), ((\sim q) \rightarrow (\sim p))$;
(*b*) $((p \vee q) \wedge r), ((p \wedge r) \vee (q \wedge r))$;
(*c*) $(((\sim p) \wedge (\sim q)) \rightarrow (\sim r)), (r \rightarrow (q \vee p))$;
(*d*) $(((\sim p) \vee q) \rightarrow r), ((p \wedge (\sim q)) \vee r)$.

7 Show that the statement form $(((\sim p) \rightarrow q) \rightarrow (p \rightarrow (\sim q)))$ is not a tautology. Find statement forms $\mathscr{A}$ and $\mathscr{B}$ such that $(((\sim \mathscr{A}) \rightarrow \mathscr{B}) \rightarrow (\mathscr{A} \rightarrow (\sim \mathscr{B})))$ is a contradiction.

1.3 Rules for manipulation and substitution

Proposition 1.9

If $\mathscr{A}$ and $(\mathscr{A} \rightarrow \mathscr{B})$ are tautologies, then $\mathscr{B}$ is a tautology.