Modern Methods in Topological Vector Spaces

ALBERT WILANSKY

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MODERN METHODS IN TOPOLOGICAL VECTOR SPACES

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PREFACE

This book, designed for a one-year course at the beginning graduate level, displays those properties of topological vector spaces which are used by researchers in classical analysis, differential and integral equations, distributions, summability, and classical Banach and Fréchet spaces. In addition, optional examples and problems (with hints and references) will set the reader's foot on numerous paths such as non-locally convex (e.g., ultrabarrelled) spaces, Köthe–Toeplitz spaces, Banach algebra, sequentially barrelled spaces, and norming subspaces.

The prerequisites are laid out in Chapter 1, which is a rapid sketch of vector spaces and point set topology. The central theme of the book is duality, which is taken up in Chapter 8. In an ideal world the course would begin with this chapter, the material of the preceding seven being known to all educated persons. The climax is reached in Chapter 12, which presents completeness theorems in this setting: a function space is complete when membership in it is secured by continuity on a certain family of sets. (See the beginning of Chapter 12.) The remaining three chapters treat special topics such as inductive limits, distributions, weak compactness, and barrelled spaces, by means of the tools developed in Chapter 12. In particular, the separable quotient problem for Banach spaces (Section 15-3), as special and as classical as it appears, requires much of this material for its fullest understanding.

The style is that of a beginning text in which concepts are explained and motivated, and every theorem is delineated by examples which show that its hypotheses are minimal and which illustrate how the theorem is used, how it fits into the theory, and how it forms a step in some general program. For example, the equivalence program is a body of results of the form $P \equiv (Q \Rightarrow R)$ where P is a property of a space and Q, R are properties of sets in its dual. [See, for example, Theorem 9-3-4(b) and (c) and the beginning of Section 9-4.] Moreover, the book is more completely cross-referenced than most others that I have seen.

Both nets and filters are introduced and used whenever appropriate.

A set of 33 tables is given at the end of the book, allowing quick reference to theorems and counterexamples. There are also 1500 problems which are arranged in four sequences at the end of each section; the few problems whose numbers are below 100 are considered part of the text. (See Section 1-1.)

REMARKS TO THE EXPERIENCED READER

A possibly unfamiliar concept is that of property of a dual pair. A dual pair (X, Y) is said to have a property P if X has a compatible topology with property P. (See Remark 8-6-8.) Thus (c_0, l) is a complete dual pair while (φ, l) is not.

The open mapping and closed graph theorems (in the primitive case) are proved without use of quotients; completeness of the dual of a bornological space is given a simple direct proof (Corollary 8-6-6); and the more sophisticated deduction from Grothendieck's theorem is given later (Example 12-2-20.) Five unique features of the book are:

1. Boundedness is proved to be a duality invariant in an easy way (Theorem 8-4-1), long before the appearance of the Banach–Mackey theorem (10-4-8). This method is due to H. Nakano.
2. Relatively strong topologies and an easy version of the Mackey–Arens theorem (8-2-14) are given before the standard identification of the Mackey topology (Theorem 9-2-3).
3. The convex compactness property and sequential completeness are emphasized and shown to have the same consequences in many cases (e.g., Theorems 10-4-8 and 10-4-11). Since bounded completeness implies both of these properties, this represents an improvement of the usual treatment.
4. F linked topologies (Definition 6-1-9) are featured, with the consequent upper heredity of all forms of completeness. This ties in with the aforementioned properties of dual pairs, e.g., Mazur (Definition 8-6-3) is downward hereditary so a dual pair (X, Y) is Mazur if and only if $\sigma(X, Y)$ is (Problem 8-6-101).
5. Emphasis is placed on converse theorems (Section 12-6). This is the "Bourbaki program" of finding the natural setting for classical Banach space theorems. This includes the first textbook appearance of a recently discovered simple proof of Mahowald's characterization of barrelled spaces (Theorem 12-6-3).

ACKNOWLEDGMENTS

I have made extensive use of earlier texts [8, 14, 26, 37, 38, 58, 82, 88, 116, 138], the book of H. H. Schaefer, and lecture notes of D. A. Edwards.

Many results in the book are attributed—others are not because they have passed into the lore of the subject. Some are unattributed by oversight; an example is Theorem 14-4-11 which I recently found out occurs in [24].

My book would have been impossible to write without constant consultation of *Mathematical Reviews*. Science owes a vast debt to the dedicated staff of this most excellent and complete publication.

A great stroke of fortune brought D. J. H. Garling, G. Bennett, and N. J. Kalton to Lehigh University for a year or two under the laughable misapprehension that they had something to learn from me. Any virtue this book may happen to possess is largely due to their generosity and patience during their visit and in subsequent correspondence.

A special word of thanks goes to J. Diestel for many fruitful conversations in bars and cocktail lounges throughout the United States. I had many occasions to consult A. K. Snyder and also received much help from E. G. Ostling, D. B. Anderson, W. G. Powell, C. L. Madden, and W. H. Ruckle.

The excellent Mathematics Department of Lehigh University provides an ambience in which scholarship and mutual assistance prevails.

My daughter Carole Wilansky helped with many organizational chores. Judy Arroyo typed the whole book with unbelievable speed and accuracy—I greatly appreciate her dedication to the project. My thanks also to the staff of McGraw-Hill for their helpfulness at every step.

Albert Wilansky,
January 1978

CONTENTS

CHAPTER

ONE

INTRODUCTION

1-1 EXPLANATORY

We use the notations of elementary set theory such as $A \subset B$ (A is a subset of B). The only possibly unfamiliar one is the useful symbol $\not\pitchfork$; $A \not\pitchfork B$ (A does not meet B) means that A, B are disjoint. When $A \subset X$, $X \backslash A$ or (when X is understood) $\tilde{A}$ is the complement of A (in X).

We use R^n, $\mathscr{C}^n$ for the spaces of n-tuples of real, respectively complex, numbers; $R = R^1$, $\mathscr{C} = \mathscr{C}^1$. Every vector space X has for its space of scalars the space $\mathscr{K}$ and, in this book, $\mathscr{K}$ is always R or $\mathscr{C}$; all statements made will be correct for either interpretation except when we specifically mention real vector space or complex vector space. (To avoid duplication, proofs are written using complex scalars.)

Problems

Those numbered from 1 to 99 are basic for further developments and form part of the text. Problems numbered >200 are more difficult, and those numbered >300 are really notes with references to the literature.

Proof Brackets

When part of a discussion is enclosed thus $[\![\ldots]\!]$ it means that the immediately preceding statement is being proved. For example, suppose that the text reads: "Since $x \neq 0$ $[\![$if $x = 0$, $\cos x = 1$, contradicting the hypotheses$]\!]$ we may cancel x." The reader should first absorb "Since $x \neq 0$, we may cancel x." He may then, if required, consult the proof in brackets.

Notation

δ^k is the sequence x where $x_k = 1$, $x_n = 0$ for $n \neq k$; that is, δ_n^k is the Kronecker delta. For $x \in \mathscr{K}$, sgn x is defined to be $|x|/x$ if $x \neq 0$; and sgn $0 = 1$.

1-2 TABLE OF SPACES

Several spaces will be used to illustrate the developments of the text. They are all vector spaces (Sec. 1-5) and each, with a few exceptions, has a distinguished real function defined on it, called *paranorm* (Sec. 2-1) or *norm* (Sec. 2-2), and is denoted by $\|x\|$, its value at x. Whenever such a sentence occurs as "show that c has a certain property," the reader may consult this table. Unless otherwise stated, the space is supposed to be endowed with its paranorm or norm.

1. Definition If f is a scalar-valued function on a set X, $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$. This is called the sup *norm.*

In particular if x is a sequence, $\|x\|_\infty = \sup\{|x_n| : n = 1, 2, \ldots\}$.

2. Definition If f is a scalar-valued function on $[0, 1]$, $\|f\|_p = (\int_0^1 |f(t)|^p\,dt)^{1/p}$ if $p \geq 1$, $\|f\|_p = \int_0^1 |f(t)|^p\,dt$ if $0 < p < 1$. For a sequence x, $\|x\|_p = (\sum |x_n|^p)^{1/p}$ if $p \geq 1$, $\|x\|_p = \sum |x_n|^p$ if $0 < p < 1$.

Table 1-2-1 Table of spaces, where all functions and sequences are scalar valued

bfa (N)	See Example 2-3-14.
bfa (H, A)	See Sec. 14-4.
$B(X, Y)$	See Definition 2-3-2.
c	Convergent sequences, with $\|x\|_\infty$.
c_0	Null sequences (i.e., converging to 0), with $\|x\|_\infty$.
$\mathscr{C}$	The complex numbers.
$C(H)$	Continuous functions on H, with $\|f\|_\infty$ if H is a compact Hausdorff space. For a general Hausdorff space, see Prob. 4-1-105.
$C^*(H)$	Bounded continuous functions on H, with $\|f\|_\infty$.
$C_0(H)$	Continuous functions vanishing at infinity (that is, $\{x \in H : \|f(x)\| \geq \varepsilon\}$ is compact for each $\varepsilon > 0$), with $\|f\|_\infty$.
cs	Convergent series, with $\|x\| = \|s\|_\infty$ where $s_n = \sum_{k=1}^n x_k$.
Disc algebra	Members of $C(D)$ which are analytic in U, with $\|f\|_\infty$ where $U = \{z \in \mathscr{C} : \|z\| < 1\}$, $D = \bar{U}$.
$\mathscr{K}$	The scalar field; either $\mathscr{C}$ or R.
l^p	The set of sequences x, with $\|x\|_p < \infty$ (Definition 1-2-2).
l^∞	The bounded sequences, with $\|x\|_\infty$.
L^p	(Equivalence classes of) measurable functions on $[0, 1]$, with $\|f\|_p < \infty$ (Definition 1-2-2).
$\mathscr{M}$	(Equivalence classes of) all measurable functions on $[0, 1]$, with $\|f\| = \int_0^1 \|f(t)\|/[1 + \|f(t)\|]\,dt$
$M(H)$	See Sec. 14-6.
m_0	See Example 2-3-15.

Table 1-2-1 *Continued*

N	The positive integers.
ω	All sequences, with $\|x\| = \sum (1/2^n)|x_n|/(1 + |x_n|)$.
φ	All finite sequences, that is, x, such that $x_n = 0$ eventually.
R	The real numbers.

1-3 SOME COMPUTATIONS

A few useful results from classical analysis are presented in this section.

Suppose that $f''(x) \geq 0$ for $x > 0$. Then, for $0 < a < x < b$,

$$\frac{f(x) - f(a)}{x - a} = \frac{1}{x - a}\int_a^x f' \leq f'(x) \leq \frac{1}{b - x}\int_x^b f' = \frac{f(b) - f(x)}{b - x}$$

Hence

$$f(x) \leq \frac{b - x}{b - a}f(a) + \frac{x - a}{b - a}f(b)$$

Apply this to the function $f = -\log$. With $\theta = (b - x)/(b - a)$ we have

$$a^\theta b^{1-\theta} \leq \theta a + (1 - \theta)b \tag{1-3-1}$$

By symmetry, (1-3-1) holds for all positive a, b and $0 \leq \theta \leq 1$.

Now with $\{a_n\}$, $\{b_n\}$ nonnegative real sequences, $A = \sum a_n$, $B = \sum b_n$, we have $\sum(a_n/A)^\theta(b_n/B)^{1-\theta} \leq (\theta/A)\sum a_n + [(1 - \theta)/B]\sum b_n = 1$, and so

$$\sum a_n^\theta b_n^{1-\theta} \leq \left(\sum a_n\right)^\theta\left(\sum b_n\right)^{1-\theta} \tag{1-3-2}$$

Let u_n, v_n be complex sequences, $p > 1$, $1/p + 1/q = 1$, and, in (1-3-2), set $\theta = 1/p$, $a_n = |u_n|^p$, $b_n = |v_n|^q$. We obtain *Hölder's inequality*:

$$\sum |u_n v_n| \leq \left(\sum |u_n|^p\right)^{1/p}\left(\sum |v_n|^q\right)^{1/q} \tag{1-3-3}$$

that is, $\|uv\|_1 \leq \|u\|^p \|v\|^q$, $1/p + 1/q = 1$.

Applying (1-3-3) to partial sums we see that convergence of the series on the right implies convergence of the left-hand series. The same remark applies to the following arguments.

For $p > 1$, $1/p + 1/q = 1$, applying (1-3-3) gives ($a_n \geq 0$, $b_n \geq 0$)

$$\begin{aligned}\sum(a_n + b_n)^p &= \sum a_n(a_n + b_n)^{p-1} + \sum b_n(a_n + b_n)^{p-1}\\ &\leq \left(\sum a_n^p\right)^{1/p}\left[\sum(a_n + b_n)^{(p-1)q}\right]^{1/q} + \left(\sum b_n^p\right)^{1/p}\left[\sum(a_n + b_n)^{(p-1)q}\right]^{1/q}\end{aligned}$$

Dividing the first and last terms by $\left[\sum(a_n + b_n)^p\right]^{1/q}$ and using $(p - 1)q = p$, we obtain

$$\left[\sum(a_n + b_n)^p\right]^{1/p} \leq \left(\sum a_n^p\right)^{1/p} + \left(\sum b_n^p\right)^{1/p}$$

and so

$$\left(\sum |u_n + v_n|^p\right)^{1/p} \leq \left(\sum |u_n|^p\right)^{1/p} + \left(\sum |v_n|^p\right)^{1/p} \qquad p \geq 1 \tag{1-3-4}$$

Now let a, b be complex numbers, set $u_1 = |a|^{1/p}$, $v_2 = |b|^{1/p}$, all other u_i and $v_j = 0$. Then $(|a| + |b|)^{1/p} \le |a|^{1/p} + |b|^{1/p}$. Thus, for $0 < p < 1$, $|a + b|^p \le |a|^p + |b|^p$ and so

$$\sum |u_n + v_n|^p \le \sum |u_n|^p + \sum |v_n|^p \qquad 0 < p < 1 \tag{1-3-5}$$

We shall refer to both (1-3-4) and (1-3-5) as *Minkowski's inequality.* Each shows that $\| u + v \|_p \le \| u \|_p + \| v \|_p$.

Next is given an important theorem proved by I. Schur in 1920. Let $A = (a_{nk})$ be a matrix of complex numbers. For $x \in \omega$, let $(Ax)_n = \sum_k a_{nk} x_k$, $Ax = \{(Ax)_n\}$, if these series converge.

1. Definition For a matrix A, $\| A \| = \sup_n \sum_k |a_{nk}|$.

This is called norm A (it may be ∞) and the reason for the notation is explained in Prob. 3-3-103. It will be seen in Remark 15-2-3 that the assumption $\| A \| < \infty$ is redundant in Theorem 1-3-2.

2. Theorem Suppose that $\| A \| < \infty$ and $Ax \in c_0$ for every sequence x of zeros and ones. Then $\sum_k |a_{nk}| \to 0$ as $n \to \infty$.

PROOF If the result is false we may assume that $\sum_k |a_{nk}| \to 1$. ⟦Choose a sequence $\{i(n)\}$ of integers such that $\sum_k |a_{i(n)k}| \to t > 0$. Let $b_{nk} = a_{i(n)k}/t$. Then $\| B \| < \infty$, $Bx \in c_0$ for each sequence x of zeros and ones and $\sum |b_{nk}| \to 1$. It is sufficient to prove that B cannot exist.⟧ It follows that

$$\lim_{n\to\infty} a_{nk} = 0 \qquad \text{for each } k$$

and

$$\lim_{n\to\infty} \sum_{k=m}^{\infty} |a_{nk}| = 1 \qquad \text{for each } m \tag{1-3-6}$$

The second part follows from the first, which is proved by setting $x = \delta^k$ in the hypothesis. Now choose $r(1)$ so that $\sum_k |a_{r(1)k}| > \frac{1}{2}$, then $m(1)$ so that $\sum_{k=1}^{m(1)} |a_{r(1)k}| > \frac{1}{2}$, and then $\sum_{k=m(1)+1}^{\infty} |a_{r(1)k}| < \frac{1}{4}$. ⟦Choose separate m's to satisfy each inequality and let $m(1)$ be the larger.⟧ Now by (1-3-6) we may choose $r(2) > r(1)$ so that $\sum_{k=1}^{m(1)} |a_{r(2)k}| < \frac{1}{8}$ and $\sum_{k=m(1)+1}^{\infty} |a_{r(2)k}| > \frac{1}{2}$. Next choose $m(2) > m(1)$ so that $\sum_{k=m(1)+1}^{m(2)} |a_{r(2)k}| > \frac{1}{2}$ and $\sum_{k=m(2)+1}^{\infty} |a_{r(2)k}| < \frac{1}{8}$. Continuing, we obtain $\sum_{k=1}^{m(i-1)} |a_{r(i)k}| < \frac{1}{8}$, $\sum_{k=m(i-1)+1}^{\infty} |a_{r(i)k}| > \frac{1}{2}$; and $\sum_{k=m(i-1)+1}^{m(i)} |a_{r(i)k}| > \frac{1}{2}$, $\sum_{k=m(i)+1}^{\infty} |a_{r(i)k}| < \frac{1}{8}$. Now define $x_k = \operatorname{sgn} a_{r(1)k}$ for $k = 1, 2, \ldots, m(1)$, $x_k = \operatorname{sgn} a_{r(2)k}$ for $k = m(1) + 1, m(1) + 2, \ldots, m(2)$, and so on. Then $x_k = \pm 1$ for each k and so $Ax \in c_0$. ⟦Let $y = (x + 1)/2$. Then y is a sequence of zeros and ones and $x = 2y - 1$.⟧ But

$$|(Ax)_{r(i)}| = \left| \sum_{k=1}^{m(i-1)} a_{r(i)k} x_k + \sum_{k=m(i-1)+1}^{m(i)} |a_{r(i)k}| + \sum_{k=m(i)+1}^{\infty} a_{r(i)k} x_k \right|$$

$$\geq \tfrac{1}{2} - \sum_{k=1}^{m(i-1)} - \sum_{k=m(i+1)}^{\infty} |a_{r(i)k}|$$

$$> \tfrac{1}{2} - \tfrac{1}{8} - \tfrac{1}{8} = \tfrac{1}{4}$$

Thus $Ax \notin c_0$.

PROBLEMS

1 If $x \in l^p$, $y \in l^q$, $1/p + 1/q = 1$, show that $\{x_n y_n\} \in l$.

---

101 Show that equality holds in (1-3-3) if and only if there exist constants A, B such that $A|u_n|^p = B|v_n|^q$.

102 Show that equality holds in (1-3-4) if and only if there exist constants A, B such that $Au_n = Bv_n$.

103 Show that (1-3-4) is false for $0 < p < 1$.

104 State and prove Hölder's and Minkowski's inequalities for integrals.

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201 Show that, for $p > 0$, $t(p) = (\sum |u_n|^p)^{1/p}$ is a decreasing function of p. [[See [153], p. 7.]]

202 Show that (Prob. 1-3-201) $\lim \{t(p): p \to +\infty\} = \inf \{t(p): p > 0\} = \max |u_n|$. [[See [153], p. 7.]]

1-4 NETS

We begin with the concept of a *partially ordered* set, abbreviated *poset*. This is a set X together with a relation $\geq$, with $x \geq y$ true or false for each $x, y \in X$. We assume that the relation is *reflexive*, that is, $x \geq x$ for all x, and *transitive*, that is, $x \geq y \geq z$ imply $x \geq z$. Some authors also require that it be *antisymmetric*, that is, $x \geq y \geq x$ imply $y = x$, but this rules out certain posets which rise naturally in convergence discussions (Prob. 1-4-1). Introducing such a relation into a set is called *ordering the set*.

Extremely important examples are the set P of all subsets of a set X with

(*a*) *order by inclusion*: $A \geq B$ means $A \subset B$;
(*b*) *order by containment*: $A \geq B$ means $A \supset B$.

A *directed set* is a poset with the additional property that for each x, y there exists z with $z \geq x$, $z \geq y$. For example, R with its usual order is a directed set.

A *chain* is a poset which is antisymmetric and satisfies $x \geq y$ or $y \geq x$ for each pair of members x, y; that is, any two members are *comparable*. For example, R is a chain.

Any subset of a poset is a poset with the same ordering and might possibly be a chain. For example, give R^2 the order $(a, b) \geq (x, y)$ means $a \geq x$ and $b \geq y$ in the ordinary sense. Then the X axis is a chain. Indeed, it is a *maximal chain* in that it is properly contained in no other chain, although there are other chains such as $\{(x, x): x \in R\}$.

We shall now state an axiom of set theory. This axiom will be an unstated hypothesis in all theorems where the phrase "let C be a maximal chain" occurs in the proof. The first such is Theorem 1-5-5.

1. Axiom: Maximal axiom Every nonempty poset includes a maximal chain.

Some references for a discussion of the place of this axiom in mathematics, and alternate forms of the axiom, may be found in [156], Sec. 7-3.

A *net* is a function defined on some directed set. For example, a sequence is a net defined on the positive integers. Just as there are sequences of points, numbers, functions, so there are nets of points, numbers, functions. For example, a net of real numbers is a function $x: D \to R$ where D is some directed set. Such a net is written $(x_\delta: D)$, and in this case x_δ is a real number for each $\delta \in D$.

Now suppose that $(x_\delta: D)$ is a net in some set X, that is, $x: D \to X$ is a map. Let $S \subset X$. We say that $x \in S$ *eventually* if there exists $\delta_0 \in D$ such that $x_\delta \in S$ for all $\delta \geq \delta_0$.

2. Example Let $D = \{\delta \in \mathscr{C}: |\delta| \leq 1\}$. Order D by $\delta \geq \delta'$ if $|\delta| \leq |\delta'|$. Let $u = (u_\delta: D)$ be the net of complex numbers given by $u_\delta = e^\delta$. Let $S = \{z \in \mathscr{C}: |z - 1| < 10^{-2}\}$. Then $u \in S$ eventually. This is just a special case of the familiar fact that $e^\delta \to 1$ as $\delta \to 0$.

We also say that a net has certain properties eventually; for example, "$|x_\delta - 2| < 1$ eventually" means "$x_\delta \in \{x: |x - 2| < 1\}$ eventually."

PROBLEMS

1 Although any two members of D, Example 1-4-2, are comparable, D is not a chain. ⟦It is not antisymmetric.⟧

2 Show that the set of all subsets of a set X ordered by inclusion is a directed set. Show the same for containment. (For this reason we use the phrase "directed by inclusion" to mean "ordered by inclusion.")

3 Show that the directed set in Prob. 1-4-2 is not a chain if X has more than one point.

4 Let D be a directed set and S a nonempty finite subset. Show that there exists x with $x \geq s$ for all $s \in S$.

5 Let X be a directed set and $U_1, U_2, \ldots, U_n$ subsets of X. Suppose that x is a net with, for each i, $x \in U_i$ eventually. Show that $x \in \cap U_i$ eventually ⟦Prob. 1-4-4⟧.

---

101 Give an example of a poset which is not a directed set.

102 The *discrete order* on a set X is defined by $x \geq y$ if and only if $x = y$. The *indiscrete order* has $x \geq y$ for all x, y. Which of these is directed? antisymmetric?

103 The set of discs in the plane, ordered by containment, is a directed set but no a lattice. (A *lattice* is an antisymmetric poset such that each pair has a least upper bound and a greatest lower bound.)

104 Show that Prob. 1-4-4 becomes false if S is allowed to be infinite.

105 Describe the ordering of names in a telephone book. This is called the *lexicographic* order. Show how to order R^n lexicographically.

106 Let $D = (0, 1)$ with the usual order. Let $f_\delta(t) = (\cos t)^{1-\delta}$ for $\delta \in D$, $t \in I$ where I is some closed interval in R. Show that $\| f_\delta \|_\infty < 10^{-2}$ eventually if $I = [\frac{1}{2}, 1]$, but not if $I = [0, 1]$.

---

201 The maximal axiom for countable posets is equivalent to induction.

1-5 VECTOR SPACE

As mentioned in Sec. 1-1, our vector spaces have the scalar field $\mathscr{K}$, which is R or $\mathscr{C}$. Throughout this section, X denotes a fixed vector space. For $A \subset X$, the *span of A* is the set of all (finite) linear combinations of A; it is a vector subspace of X. For a vector subspace S and point x, $S + [x]$ denotes the span of $S \cup \{x\}$. If the span of A is equal to X we say that *A spans X*.

A subset $A \subset X$ is called *convex* if $sA + tA \subset A$ for $0 \le t \le 1$, $s + t = 1$; *balanced* if $tA \subset A$ for $|t| \le 1$; and *absorbing* if for every $x \in X$ there exists $\varepsilon > 0$ such that $tx \in A$ for $|t| < \varepsilon$. A vector subspace S of X is called maximal if $S \ne X$ and $X = S + [x]$ for some x.

A function $f: X \to \mathscr{K}$ is called a *functional* and X^* denotes the vector space of all *linear* functionals on X, that is, those satisfying $f(sx + ty) = sf(x) + tf(y)$ for $s, t \in \mathscr{K}$, $x, y \in X$.

There is a natural correspondence between linear functionals and maximal subspaces as follows. *For each nonzero $f \in X^*$, $f^\perp = \{x : f(x) = 0\}$ is a maximal subspace. For each maximal subspace S there exist many $f \in X^*$ such that $f^\perp = S$ but only one whose value at any specified $a \notin S$ is a specified nonzero scalar u.* ⟦Fix $a \notin f^\perp$. Then $x - [f(x)/f(a)]a \in f^\perp$. Conversely, fix $a \notin S$. Every x is $s + ta$, $s \in S$, and we may set $f(x) = tu$.⟧

1. Theorem Let $f, f_1, f_2, \ldots, f_n \in X^*$ and $f^\perp \supset \cap\{f_i^\perp : i = 1, 2, \ldots, n\}$. Then $f = \sum t_i f_i$.

PROOF For $n = 1$, write $f_1 = g$. We may assume $g \ne 0$. Say $g(a) = 1$. Then for each x, $x - g(x)a \in g^\perp \subset f^\perp$ so $0 = f(x) - g(x)f(a)$. Thus $f = [f(a)]g$. Proceeding by induction, let $f^\perp \supset \cap\{f_i^\perp : i = 1, 2, \ldots, n + 1\}$. Let $g = f \,|\, f_{n+1}^\perp$ (the *restriction* of f to the smaller subspace), $g_i = f_i | f_{n+1}^\perp$ for $i = 1, 2, \ldots, n$. Then $g^\perp \supset \cap\{g_i^\perp : i = 1, 2, \ldots, n\}$. By the induction hypothesis $g = \sum t_i g_i$ and so $f(x) = \sum_{i=1}^n t_i f_i(x)$ for $x \in f_{n+1}^\perp$. By the case $n = 1$, this implies that $f - \sum_{i=1}^n t_i f_i = t f_{n+1}$.

Now let X be a complex vector space and X_R the same space but using only real scalars; thus X_R is a real vector space. Let Rz denote the real part of the complex number z.

2. Theorem Let X be a complex vector space, $f \in X^*$. Then $Rf \in (X_R)^*$. Moreover, for each $g \in (X_R)^*$ there exists unique $f \in X^*$ such that $g = Rf$.

PROOF The first part is trivial. Next, to prove uniqueness, let $g \in (X_R)^*$, $f \in X^*$ with $g = Rf$. Write $f = g + ih$, $h \in (X_R)^*$. Then $g(ix) + ih(ix) = f(ix) = if(x) = ig(x) - h(x)$. Equating real parts yields $h(x) = -g(ix)$ and so h, hence f, is uniquely determined if it exists. Finally, given g, define $h \in (X_R)^*$ by $h(x) = -g(ix)$ (the only formula that could work!). Let $f = g + ih$, and we shall prove that f is linear. It is clear that $f(x+y) = f(x) + f(y)$ and $f(tx) = tf(x)$ for real t; but also $f(ix) = g(ix) + ih(ix) = g(ix) - ig(iix) = g(ix) + ig(x) = i[g(x) - ig(ix)] = if(x)$.

3. Definition A Hamel basis for X is a linearly independent set which spans X.

An *n-dimensional space* ($n < \infty$) is one which has a Hamel basis with n members.

4. Theorem Let X be an n-dimensional vector space, $n < \infty$. Then X^* is also n-dimensional. Further, for each $F \in X^{**}$ there exists $x \in X$ such that $F(u) = u(x)$ for all $u \in X^*$.

PROOF Since X is isomorphic with $\mathscr{K}^n$ we may as well take $X = \mathscr{K}^n$. Let $P_i \in X^*$ be defined by $P_i(x) = x_i$ for $i = 1, 2, \ldots, n$. For each $f \in X^*$, $x \in X$, we have $f(x) = f(\sum x_k \delta^k) = \sum f(\delta^k) P_k(x)$. Thus $f = \sum f(\delta^k) P_k$ and so $P = (P_1, P_2, \ldots, P_n)$ spans X^*. It is also linearly independent since if $\sum t_k P_k = 0$, for any i, $0 = \sum t_k P_k(\delta^i) = t_i$.

Next, given $F \in X^{**}$, let $x = [F(P_1), F(P_2), \ldots, F(P_n)] \in X$. Then for each $u \in X^*$, $u = \sum t_k P_k$ and so $F(u) = \sum t_k F(P_k) = \sum t_k x_k = \sum t_k P_k(x) = u(x)$.

5. Theorem Every vector space X has a Hamel basis.

PROOF Let P be the family of linearly independent subsets of X; order P by containment and let C be a maximal chain in P. (See Remark 1-5-6.) Let H be the union of the sets in C. This is the required basis. First, H is linearly independent. [This is the same as saying that every finite subset is linearly independent. But such a subset is contained in some $S \in C$ since C is a chain; hence it is linearly independent.] Also, H spans X. [H is maximal among linearly independent sets since, if not, a larger one could be adjoined to C contradicting its maximality. So for any $x \notin H$, $H \cup \{x\}$ is linearly dependent, that is, $tx + \sum t_k h_k = 0$ for some $t, t_1, t_2, \ldots, t_n$ not all 0, $h_i \in H$. Since H is linearly independent, $t \neq 0$ and so we can solve for x.]

6. Remark If $X = \{0\}$, the set P in Theorem 1-5-5 is empty (voiding use of the maximal axiom) unless we make some special convention. The one usually chosen is to say that ϕ, the empty set, is linearly independent, and span $\phi = \{0\}$. Thus X has a Hamel basis with no members, and is 0-dimensional. Hopefully, all the results given in this book are true in such special cases, but we shall not take the space to spell them out.