DISCRETE MATHEMATICS

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Discrete Mathematics

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Preface

Goals of a Discrete Mathematics Course

There are two types of courses in discrete mathematics, which have been adopted almost universally. Almost every university has introduced a one-semester course at the freshman-sophomore level. In many universities, an upper-division course has also been introduced.

The elementary course serves the mathematical needs of computer science majors and mathematics majors, many of whom take computer science courses. The combined requirements of mathematics and computer science have created curricular difficulties, which are particularly severe in sophomore year. The discrete mathematics course is looked to more and more as a resource for dealing with them. One such case: Computer science students often do not have room in their curriculum for a logic course, so the learn it in discrete math.

In the discrete mathematics course, the student learns essential topics: induction and recursion, combinatorics, graph theory, and proofs and logic. Some of these topics are treated in more depth in the upper-division course.

This text is fully adequate for all discrete mathematics courses. It serves as a text for the freshman-sophomore course and also as a text for the yearlong upper-division course.

Our Text

This text developed from the "Blue Notes" (so named due to their blue cover) that were used for years as the text for an elementary discrete mathematics course at Seton Hall.

Starting with these notes, we began writing because we saw a need for a new discrete mathematics textbook that would give students a sense of the historical development and conceptual unity of the subject. Years of classroom experience shape our treatment of induction, recursion, proof techniques, counting, and graph theory.

This is the result of our efforts: a book that serves the mathematical needs of students taking computer science courses, with a historical perspective, and a logical, yet flexible, organization of topics.

A Text Informed by Computer Science

While this is not a computer science text, we believe that a discrete mathematics text should be informed by computer science. Two of us are computer scientists. In writing this book, we have kept in mind the needs of computer science students, and mathematics students who will take computer science courses.

The Future of Discrete Math

Discrete mathematics is a developing subject, and students should be given a view of its future. With this in mind, we discuss new ideas in our book. For example, throughout the book we discuss several new ideas about genetic code. In Chapter 2.3, in the application in 5.2, and throughout Chapter 10, students have the opportunity to read about and apply discrete mathematics to this current issue.

A Flexible Organization

The first six chapters are designed as a highly flexible text for a one-semester course in discrete mathematics at the freshman-sophomore level. The topics in these six chapters are covered in a way that allows the instructor to choose the emphasis of the course.

At the beginning of Chapter 7 the level of the text rises slightly. The last four chapters can be used as a text for a one-semester upper-division course or, for students with less preparation, for a yearlong upper-division course that begins with material from the first six chapters.

Additionally, by omitting some proofs, material from the last four chapters can be used in a course at the freshman-sophomore level. For example, we have successfully taught the material on Polya's theorem in Section 9.3 to Seton Hall freshmen.

Features

Comprehensive Coverage of Topics We cover an unusually wide set of topics. Our treatment of combinatorics, such a basic topic, is more thorough than the treatment in comparable texts. With an eye towards the future of discrete mathematics, we have also chosen to cover topics (e.g. lattices) that are expected to be very important in the years ahead.

Our treatment of proof techniques is more thorough than many texts in this market. We initially concentrate on induction and recursion, but choose to cover proof techniques and logic later in the book, when we feel that students are prepared better.

Integrated Historical Perspective The historical material is fully integrated in our text, not separated in boxes or sidelined. Reading other texts, students often have the impression that the subject is artificial or unnatural. Our book corrects this impression. Many interesting examples, not found in other texts, are included and these examples have great value in themselves, and will appeal to instructors and students alike.

Carefully Graduated Exercises The exercise sets, found at the end of each section, include Exercises, Advanced Exercises, and Computer Exercises. With a graduated level of difficulty, these high-quality exercises enable students to practice their skills as they work up to more difficult exercises.

Basic and Unique Computer Exercises The computer exercises are excellent for both math and computer science majors, adding to the foundation of computer science knowledge. We have been careful to cover standard topics, but we have also included computer exercises that are not found in other texts. Our exercises on the 3X+1 problem in Section 1.5 are an example, as are our exercises on Godel numbers in Section 10.3.

Frequent, Necessary Applications Each section ends with the description of a significant real-world application of the ideas and techniques of that section. Many topics in computer science are discussed. In the first chapter, the student is introduced to the Tower of Hanoi, the Knapsack Problem, error-correcting codes, and shift registers. In later chapters, we discuss structural induction, register allocation, network reliability, and much more.

Chapter Review At the end of each chapter there is a chapter summary with supplementary exercises, which help students both review and practice what they have learned in the preceding sections.

Guide to Literature A great resource for students, this appendix is organized by topic and lists additional resources that students can call upon to gain further understanding and insight in a particular area.

ANCILLARIES

Students' Solutions Manual—This manual offers students further help in learning and practicing their skills by including full solutions to the odd Exercises, Advanced Exercises, and Computer Exercises. (ISBN: 0-201-61924-5)

Instructor's Solutions Manual—This manual includes full solutions to the odd and even Exercises, Advanced Exercises, and Computer Exercises. (ISBN: 0-201-61924-5)

ACKNOWLEDGMENTS

This book grew out of a long and successful relationship with our students at Seton Hall, and we have dedicated it to them. To thank them would hardly be enough: without them, our book would not exist.

Among our students we particularly want to thank Joe DeVito for his hard work and enthusiasm. Joe's program gave us new insight into the 3X+1 problem. We would like to express our thanks and appreciation to the reviewers who gave us their time and advice:

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Over the years our project received strong and consistent support from our colleagues at Seton Hall. We particularly want to thank John Saccoman for his encouragement. Saccoman remarked to one of us that the text for the discrete mathematics course we were teaching might develop into a book: He was right. Dan Gross, Esther Guerin, John Masterson, Laura Schoppmann, and others in our department gave us advice and help.

Our deans, Jerry Hirsch and Jim VanOosting, looked with a benevolent eye on a project that sometimes seemed as if it would never be finished.

Eberhard Grosse, the university printer at Seton Hall, printed the Blue Notes and many versions of the text for our book.

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Neil Sloane generously shared ideas about codes with us. We learned from Joseph Brennan's incisive views about combinatorics.

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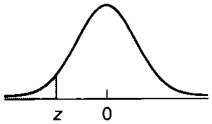
We thank Matthew Arnold and Atanas Rountev for their work on our Computer Exercises. We thank Frank Purcell for his work on the Exercises and Advanced Exercises.

The editorial staff at Addison Wesley Longman supported us with imagination and unvarying professionalism.

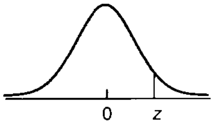
NORMAL TABLE
Areas under the
standard normal curve



	Second decimal place in z									
z	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.00
−3.9										0.0000†
−3.8	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
−3.7	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
−3.6	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0002	0.0002
−3.5	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002
−3.4	0.0002	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003
−3.3	0.0003	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0005	0.0005	0.0005
−3.2	0.0005	0.0005	0.0005	0.0006	0.0006	0.0006	0.0006	0.0006	0.0007	0.0007
−3.1	0.0007	0.0007	0.0008	0.0008	0.0008	0.0008	0.0009	0.0009	0.0009	0.0010
−3.0	0.0010	0.0010	0.0011	0.0011	0.0011	0.0012	0.0012	0.0013	0.0013	0.0013
−2.9	0.0014	0.0014	0.0015	0.0015	0.0016	0.0016	0.0017	0.0018	0.0018	0.0019
−2.8	0.0019	0.0020	0.0021	0.0021	0.0022	0.0023	0.0023	0.0024	0.0025	0.0026
−2.7	0.0026	0.0027	0.0028	0.0029	0.0030	0.0031	0.0032	0.0033	0.0034	0.0035
−2.6	0.0036	0.0037	0.0038	0.0039	0.0040	0.0041	0.0043	0.0044	0.0045	0.0047
−2.5	0.0048	0.0049	0.0051	0.0052	0.0054	0.0055	0.0057	0.0059	0.0060	0.0062
−2.4	0.0064	0.0066	0.0068	0.0069	0.0071	0.0073	0.0075	0.0078	0.0080	0.0082
−2.3	0.0084	0.0087	0.0089	0.0091	0.0094	0.0096	0.0099	0.0102	0.0104	0.0107
−2.2	0.0110	0.0113	0.0116	0.0119	0.0122	0.0125	0.0129	0.0132	0.0136	0.0139
−2.1	0.0143	0.0146	0.0150	0.0154	0.0158	0.0162	0.0166	0.0170	0.0174	0.0179
−2.0	0.0183	0.0188	0.0192	0.0197	0.0202	0.0207	0.0212	0.0217	0.0222	0.0228
−1.9	0.0233	0.0239	0.0244	0.0250	0.0256	0.0262	0.0268	0.0274	0.0281	0.0287
−1.8	0.0294	0.0301	0.0307	0.0314	0.0322	0.0329	0.0336	0.0344	0.0351	0.0359
−1.7	0.0367	0.0375	0.0384	0.0392	0.0401	0.0409	0.0418	0.0427	0.0436	0.0446
−1.6	0.0455	0.0465	0.0475	0.0485	0.0495	0.0505	0.0516	0.0526	0.0537	0.0548
−1.5	0.0559	0.0571	0.0582	0.0594	0.0606	0.0618	0.0630	0.0643	0.0655	0.0668
−1.4	0.0681	0.0694	0.0708	0.0721	0.0735	0.0749	0.0764	0.0778	0.0793	0.0808
−1.3	0.0823	0.0838	0.0853	0.0869	0.0885	0.0901	0.0918	0.0934	0.0951	0.0968
−1.2	0.0985	0.1003	0.1020	0.1038	0.1056	0.1075	0.1093	0.1112	0.1131	0.1151
−1.1	0.1170	0.1190	0.1210	0.1230	0.1251	0.1271	0.1292	0.1314	0.1335	0.1357
−1.0	0.1379	0.1401	0.1423	0.1446	0.1469	0.1492	0.1515	0.1539	0.1562	0.1587
−0.9	0.1611	0.1635	0.1660	0.1685	0.1711	0.1736	0.1762	0.1788	0.1814	0.1841
−0.8	0.1867	0.1894	0.1922	0.1949	0.1977	0.2005	0.2033	0.2061	0.2090	0.2119
−0.7	0.2148	0.2177	0.2206	0.2236	0.2266	0.2296	0.2327	0.2358	0.2389	0.2420
−0.6	0.2451	0.2483	0.2514	0.2546	0.2578	0.2611	0.2643	0.2676	0.2709	0.2743
−0.5	0.2776	0.2810	0.2843	0.2877	0.2912	0.2946	0.2981	0.3015	0.3050	0.3085
−0.4	0.3121	0.3156	0.3192	0.3228	0.3264	0.3300	0.3336	0.3372	0.3409	0.3446
−0.3	0.3483	0.3520	0.3557	0.3594	0.3632	0.3669	0.3707	0.3745	0.3783	0.3821
−0.2	0.3859	0.3897	0.3936	0.3974	0.4013	0.4052	0.4090	0.4129	0.4168	0.4207
−0.1	0.4247	0.4286	0.4325	0.4364	0.4404	0.4443	0.4483	0.4522	0.4562	0.4602
−0.0	0.4641	0.4681	0.4721	0.4761	0.4801	0.4840	0.4880	0.4920	0.4960	0.5000

† For $z \leq -3.90$, the areas are 0.0000 to four decimal places.

NORMAL TABLE (cont.)
Areas under the standard normal curve



	Second decimal place in z									
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000†									

† For $z \geq 3.90$, the areas are 1.0000 to four decimal places.

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sets, also known as *finite additivity*. The probability function also obeys the Principle of Inclusion-Exclusion for any finite number of sets.

DEFINITION 9.2.4 Two events E and F are

i. *disjoint*, if $E \cap F = \emptyset$.
ii. *independent*, if $P(E \cap F) = P(E)P(F)$.

THEOREM 9.2.1 If E and F are disjoint events, then

(i) $P(E \cup F) = P(E) + P(F)$.

It is always true that

(ii) $P(E^c) = 1 - P(E)$.

Proof (i) follows by finite additivity, since $P(E \cap F) = P(\emptyset) = 0$. (ii) follows because $P(E \cup E^c) = P(S) = 1$ and $P(E \cap E^c) = P(\emptyset) = 0$. ●

The Binomial Distribution Suppose that we toss a coin n times and that the outcomes of each toss are independent (the outcome of each toss does not affect the outcome of any other toss). The probability of getting heads is $P(H) = p$ and the probability of getting tails is $P(T) = q$, where $0 \le p, q \le 1$, and $p + q = 1$. Let X be the number of heads that comes up. Then

$$P(X = k) = \binom{n}{k} p^k q^{n-k}$$

for $k = 0, 1, \ldots, n$.

The coin is fair (it is equally likely that heads and tails come up) if $p = q = 1/2$. For a fair coin, the expression of the binomial distribution can be simplified somewhat:

$$P(X = k) = \binom{n}{k}(1/2)^n = \binom{n}{k}/2^n$$

EXAMPLE 5 A fair coin is tossed seven times, and X is the number of heads. Find the following probabilities:

(i) $P(X = 3)$.
(ii) $P(X \le 2)$.
(iii) $P(3 \le X \le 5)$.

Solution

(i) $P(X = 3) = \binom{7}{3}/2^7 = 35/128 = .2734 = 27.34\%$.
(ii) $P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2) =$
$\left(\binom{7}{0} + \binom{7}{1} + \binom{7}{2}\right)/2^7 = (1 + 7 + 21)/128 = .2266 = 22.66\%$.

(iii) $P(3 \leq X \leq 5) = P(X = 3) + P(X = 4) + P(X = 5) =$
$\left(\binom{7}{3} + \binom{7}{4} + \binom{7}{5}\right)/2^7 = (35 + 35 + 21)/128 = .7109 = 71.09\%.$ ●

Now we will give two examples to illustrate the behavior of the binomial distribution in the case where $p \neq q$.

In the summer of 1653, the Duke of Roannez introduced Pascal to Antoine Gombaud, Chevalier de Méré. De Méré was a member of the nobility and a gambler, and he asked Pascal the following question.

De Méré's Paradox Player A and Player B are playing the following game with dice. Player A tosses four dice and tries to get at least one six, while Player B tosses twenty-four pairs of dice and tries to get at least one pair of sixes. Since $4/6 = 24/36$ the game should be fair. However, experience shows that Player A has an advantage. Why does this happen?

First notice that if the dice are fair, the probability of getting a six is $p = 1/6$, while the probability of not getting a six is $q = 5/6$. We can use the rule for complements, and the binomial distribution, to compute and compare the probabilities that the two players succeed.

Pascal answered de Méré's question as follows. This was one of the first applications of the binomial distribution.

$$P(A \text{ succeeds}) = 1 - P(A \text{ fails}) = 1 - (5/6)^4 = .5177$$

while

$$P(B \text{ succeeds}) = 1 - P(B \text{ fails}) = 1 - (35/36)^{24} = .4914$$

In other words, Player A has an advantage of about 2.6%. This agrees very well with de Méré's observation.

Shown in Table 9.1 is a dice game of the kind which occurs in de Méré's Paradox. Player A tosses n dice and tries to get at least one six, while Player B tosses $6n$ pairs of dice and tries to get at least one pair of sixes. Notice that Player A always has an advantage.

Samuel Pepys, the author of the famous *Diary*, asked Isaac Newton a question about a gambling game.

Mr. Pepys to Mr. Isaac Newton, Wednesday, November 22, 1693:

> *Sir: ... Now so it is, that the late project (of which you cannot but heard) of Mr. Neale the Groom-Porter his lottery, has almost extinguished for some time at all places of public conversation in this town, especially among men of numbers, every other talk but what relates to the doctrine of determining between the true proportions of the hazards incident to this or that given chance or lot.*

Pepys went on to ask Newton the following question. Suppose that Player A tosses six dice and tries to get at least one six, while Player B tosses twelve dice and tries to get at least two sixes. Is this a fair game? Newton replied to Pepys:

Table 9.1 A Table for de Méré's Paradox

Player A tosses n dice	*Player A's Chance*	*Player B's Chance*
$n = 1$	.1667	.1555
$n = 2$	.3056	.2868
$n = 3$	.4213	.3977
$n = 4$	.5177	.4914
$n = 5$	.5981	.5705
$n = 6$	.6651	.6372
$n = 7$	.7209	.6937
$n = 8$	.7674	.7413
$n = 9$	.8062	.7816

Mr. Isaac Newton to Mr. Pepys, Cambridge, November 26, 1693:

> *Sir: I was very glad to hear of your good health by Mr. Smith, and to have any opportunity given me of showing how ready I should be to serve you or your friends upon any occasion, and wish that something of greater moment would give me a new opportunity of doing it so as to become more useful to you than in solving only a mathematical question.*

Newton explained that the game is not fair. We shall give his argument restated in modern language.

We shall analyze this game just as we analyzed de Méré's Paradox, by computing the probabilities that each player succeeds.

$$P(A \text{ succeeds}) = 1 - P(A \text{ fails}) = 1 - (5/6)^6 = .6651.$$

Remember that Player B can fail in two ways: by getting no sixes or only one six.

$$P(B \text{ succeeds}) = 1 - P(B \text{ fails})$$
$$= 1 - (5/6)^{12} - 12 \cdot (1/6)(5/6)^{11} = 1 - .1122 - .2692 = .6186$$

In other words, Player A has an advantage of more than 4.5%, a very large advantage in a gambling game. As Newton put it,

> *I say, that A has an easier task than B.*

Table 9.2 shows the Dice Game of Newton and Pepys. Player A tosses n dice and tries to get at least one six, while Player B tosses $2n$ dice and tries to get at least two sixes.

Notice how the probabilities in this game behave differently from those in the game based on de Méré's Paradox. At first, Player A has an advantage, but as more dice are tossed, the advantage shifts to Player B.

Remember that a random variable X is a function on the sample space, which we assume to be finite. Every random variable has a *distribution*

$$f(x) = P(X = x)$$