Modern Theory of
Critical Phenomena

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EDITOR'S FOREWORD

The problem of communicating in a coherent fashion the recent developments in the most exciting and active fields of physics seems particularly pressing today. The enormous growth in the number of physicists has tended to make the familiar channels of communication considerably less effective. It has become increasingly difficult for experts in a given field to keep up with the current literature; the novice can only be confused. What is needed is both a consistent account of a field and the presentation of a definite "point of view" concerning it. Formal monographs cannot meet such a need in a rapidly developing field, and, perhaps more important, the review article seems to have fallen into disfavor. Indeed, it would seem that the people most actively engaged in developing a given field are the people least likely to write at length about it.

FRONTIERS IN PHYSICS has been conceived in an effort to improve the situation in several ways. Leading physicists today frequently give a series of lectures, a graduate seminar, or a graduate course in their special fields of interest. Such lectures serve to summarize the present status of a rapidly developing field and may well constitute the only coherent account available at the time. Often, notes on lectures exist (prepared by the lecturer himself, by graduate students, or by postdoctoral fellows) and are distributed in mimeographed form on a limited basis. One of the principal purposes of the FRONTIERS IN PHYSICS Series is to make such notes available to a wider audience of physicists.

It should be emphasized that lecture notes are necessarily rough and informal, both in style and content; and those in the series will prove no exception. This is as it should be. The point of the series is to offer new, rapid, more informal, and, it is hoped, more effective ways for physicists to teach one another. The point is lost if only elegant notes qualify.

The publication of collections of reprints of recent articles in very active fields of physics will improve communication. Such collection are themselves useful to people working in the field. The value of the reprints will, however, be enhanced if the collection is accompanied by an introduction of moderate length which will serve to tie the collection together and, necessarily, constitute a brief survey of the present status of the field. Again, it is appropriate that such an introduction be informal, in keeping with the active character of the field.

The informal monograph, representing an intermediate step between lecture notes and formal monographs, offers an author the opportunity to present his views of a field which has developed to the point where a summation might prove extraordinarily fruitful but a formal monograph might not be feasible or desirable.

Contemporary classics constitute a particularly valuable approach to the teaching and learning of physics today. Here one thinks of fields that lie at the heart of much of present-day research, but whose essentials are by now well understood, such as quantum electrodynamics or magnetic resonance. In such fields some of the best pedagogical material is not readily available, either because it consists of papers long out of print or lectures that have never been published.

The above words, written in August, 1961, continue to be applicable. During the past decade, the study of critical phenomena has emerged as a major sub-field of condensed matter physics. The modern theory of critical phenomena, which is so lucidly described by Professor Ma in this volume, is based on the renormalization group approach to scaling, which was developed by Kenneth Wilson some five years ago. As Ma emphasizes, while the renormalization group approach has been extraordinarily successful in explaining a wide variety of experimental phenomena, it still lacks a firm mathematical foundation; he therefore makes a special effort to call to the attention of the reader the ambiguities and uncertainties in the theory, as well as the tentative nature of many of its conclusions.

An important contributor himself to our current understanding of critical phenomena, Professor Ma in this volume has made a major effort to introduce the fundamental theoretical concepts, i.e., mean field theory, the scaling hypothesis, and the renormalization group approach, to the beginner, to the graduate student who has no previous knowledge of critical phenomena. He then goes on to apply the renormalization group to selected problems, with emphasis on the underlying physics and the basic assumptions involved.

I share his hope that this volume may be used with profit as an introductory text in a course addressed to first- or second-year graduate students, as well as being of use in an advanced course in statistical physics.

DAVID PINES

March 1976

PREFACE

This book is an introduction to the modern theory of critical phenomena. Research in this field has been extensive over the past few years and the theory has undergone rapid development following the pioneering work of Wilson (1971) on the renormalization group (abbreviated as RG) approach to scaling.

This book is intended for use by the graduate student in physical sciences or engineering who has no previous knowledge of critical phenomena. Nor does the reader need any background in group theory or any advanced mathematics. Elementary statistical mechanics is the only prerequisite.

The first six chapters can be read without going through any tedious calculation and can be used as an introductory text. They cover the outstanding features of critical phenomena and basic ideas of the RG. The rest of the book treats more advanced topics, and can be used in an advanced course in statistical physics.

I want to stress the distinction between the following two approaches to complex physical problems:

(i) Direct solution approach. This means calculation of physical quantities of interest in terms of parameters given in the particular model — in other words, solving the model. The calculation may be done analytically or numerically, exactly or approximately.

(ii) Exploiting symmetries. This approach does not attempt to solve the model. It considers how parameters in the model change under certain symmetry transformations. From various symmetry properties, one deduces some characteristics of physical quantities. These characteristics are generally independent of the quantitative values of the parameters. By symmetry transformations I mean those which are relatively simple, like reflection, translation, or rotation. I would not call a complete solution of a complicated model a symmetry property of that model.

Approach (ii) is not a substitute for approach (i). Experience tells us that one should try (ii) as far as one can before attempting (i), since (i) is often a very difficult task. Results of (ii) may simplify this task greatly. Outstanding examples of this may be found in the study of rotations in atomic physics, translations in solid state physics, and isotopic spin rotations in nuclear physics. A great deal can be learned from (ii) even without attempting (i).

To a large extent, the traditional effort in the theory of critical phenomena has taken approach (i). The mean field theory is an example of an approximate solution. Onsager's theory of the Ising model is an example of an exact solution. There are many numerical solutions of various models. While the mean field theory often seems too crude, the exact solutions are too complicated. A peculiar feature of critical phenomena is that there is very little one can do to improve the mean field theory substantially without solving the problem exactly. This makes the theory of critical phenomena a very difficult field. Many of its contributors have been mathematical talents.

The new renormalization group theory takes approach (ii). The renormalization group is a set of symmetry transformations. It tells a great deal when applied to critical phenomena although it is not a substitute for a complete solution. I would say that its role in critical phenomena is as important as the role of the rotation group in atomic physics. Although it is not as simply defined as rotations, it is not too complicated either. The fact that it is accessible to mathematically less sophisticated people like myself is an important reason for the recent rapid advances in critical phenomena. The field is now less exclusive, so that many can now understand and contribute to it.

The purpose of this volume is to introduce this new approach, beginning at a very elementary level, and to present a few selected topics in some detail. Some technical points which are often taken for granted in the literature are elaborated. This volume is not intended to be a review of the vast new field, but rather to serve as a text for those who want to learn the basic material and to equip themselves for more advanced readings and contributions.

In spite of its great success, the new renormalization group approach to the theory of critical phenomena still lacks a firm mathematical foundation. Many conclusions still remain tentative and much has not been understood. It is very important to distinguish between plausible hypotheses and established facts. Often the suspicious beginner sees this distinction very clearly. However, after he enters the field, he is overwhelmed by jargon and blinded by the successes reported in the literature. In this volume the reader will encounter frequent emphasis upon ambiguities and uncertainties. These emphases must not be interpreted as discouraging notes, but are there simply to remind the reader of some of the questions which need to be resolved and must not be ignored.

The book is roughly divided into two parts. The first part is devoted to the elaboration of basic ideas following a brief survey of some observed critical phenomena. The second part gives selected applications and discussion of some more technical points.

Very little will be said about the vast literature concerned with approach (i) mentioned above, since there are already many books and reviews available and our main concern is approach (ii) via the RG. However, the mean field theory and the closely related Gaussian approximation will be discussed in detail (Chapter III) because they are very simple and illustrative.

Kadanoff's idea of block construction (1966) will be introduced at an early stage (Chapter II), as it is an essential ingredient of RG theory. The scaling hypothesis will be introduced as a purely phenomenological hypothesis (Chapter IV). The idea of scale transformations is also fundamental to the RG. The definitions of the RG, the idea of fixed points, and connection to critical exponent will be examined in Chapters V and VI.

The basic abstract ideas of the RG are easy to understand, but to carry out these ideas and verify them explicitly turns out to be difficult. Even the simplest examples of the realization of the RG are rather complicated. Several examples, including Wilson's approximate recursion formula, the case of small ϵ, and some two-dimensional numerical calculations, will be presented, and some fundamental difficulties and uncertainties discussed (Chapters VII, VIII).

The very successful technique of the ϵ expansion and the $1/n$ expansion will be developed and illustrated with simple calculations. The basic assumptions behind these expansions are emphasized (Chapter IX). The effect of impurities on critical behaviors will be discussed at length, followed by a study of the self-avoiding random walk problem (Chapter X).

The material in the first ten chapters concerns static (time-averaged) critical phenomena. The remaining four chapters will be devoted to dynamic (time-varying) critical phenomena. Mode-mode coupling, relaxation times, the generalization of the RG ideas to dynamics, etc. will be explained (Chapters XI, XII). A few simple dynamic models are then discussed as illustrations of the application of the RG ideas (Chapter XIII). Finally the perturbation expansion in dynamics is developed and some technical points are elaborated (Chapter XIV).

The material presented in this volume covers only a small fraction of the new developments in critical phenomena over the past four years. Instead of briefly discussing many topics, I have chosen to discuss a few topics in some depth. Since I came to the study of critical phenomena and the RG only quite recently, I remember well the questions a beginner asks, and have tried throughout this volume to bring up such questions and to provide answers to them. Many of the questions brought up, however, still have no answer.

My knowledge in this field owes much to my collaboration and conversations with several colleagues, A. Aharony, M. E. Fisher, B. I. Halperin, P. C. Hohenberg, Y. Imry, T. C. Lubensky, G. F. Mazenko, M. Nauenberg, B. G. Nickel, P. Pfeuty, J. C. Wheeler, K. G. Wilson, and Y. Yang. I am very grateful to K. Friedman, H. Gould, G. F. Mazenko, W. L. McMillan, J. Rehr, A. Aharony, K. Elinger and J. C. Wheeler for their valuable comments on the manuscript.

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