Integral Transforms in Science and Engineering

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Preface

Integral transforms are among the main mathematical methods for the solution of equations describing physical systems, because, quite generally, the coupling between the elements which constitute such a system—these can be the mass points in a finite spring lattice or the continuum of a diffusive or elastic medium—prevents a straightforward "single-particle" solution. By describing the same system in an appropriate reference frame, one can often bring about a mathematical *uncoupling* of the equations in such a way that the solution becomes that of noninteracting constituents. The "tilt" in the reference frame is a finite or integral transform, according to whether the system has a finite or infinite number of elements. The types of coupling which yield to the integral transform method include diffusive and elastic interactions in "classical" systems as well as the more common quantum-mechanical potentials.

The purpose of this volume is to present an orderly exposition of the theory and some of the applications of the finite and integral transforms associated with the names of Fourier, Bessel, Laplace, Hankel, Gauss, Bargmann, and several others in the same vein.

The volume is divided into four parts dealing, respectively, with finite, series, integral, and canonical transforms. They are intended to serve as independent units. The reader is assumed to have greater mathematical sophistication in the later parts, though.

Part I, which deals with finite transforms, covers the field of complex vector analysis with emphasis on particular linear operators, their eigenvectors, and their eigenvalues. Finite transforms apply naturally to lattice structures such as (finite) crystals, electric networks, and finite signal sets.

Fourier and Bessel series are treated in Part II. The basic theorems are proven here in the customary classical analysis framework, but when introducing the Dirac δ, we do not hesitate in translating the vector space concepts from their finite-dimensional counterparts, aiming for the rigor of most mathematical physics developments. The appropriate warning signs are placed where one is bound, nevertheless, to be led astray by finite-dimensional analogues. Applications include diffusive and elastic media of finite extent and infinite lattices.

Fourier transforms occupy the major portion of Part III. After their introduction and the study of their main properties, we turn to the treatment of certain special functions which have close connection with Fourier transforms and which are, moreover, of considerable physical interest: the attractive and repulsive quantum oscillator wave functions and coherent states. Other integral transforms (Laplace, Mellin, Hankel, etc.) related to the Fourier transform and applications occupy the rest of this part.

"Canonical transforms" is the name of a parametrized continuum of transforms which include, as particular cases, most of the integral transforms of Part III. They also include Bargmann transforms, a rather modern tool used for the description of shell-model nuclear physics and second-quantized boson field theories. In the presentation given in Part IV, we are adapting recent research material such as canonical transformations in quantum mechanics, hyperdifferential operator realizations for the transforms, and similarity groups for a class of differential equations. We do not explicitly use Lie group theory, although the applications we present in the study of the diffusion and related Schrödinger equations should cater to the taste of the connoisseur.

On the whole, the pace and tone of the text have been set by the balance of intuition and rigor as practiced in applied mathematics with the aim that the contents should be useful for senior undergraduate and graduate students in the scientific and technical fields. Each part contains a flux diagram showing the logical concatenation of the sections so as to facilitate their use in a variety of courses. The graduate student or research worker may be interested in some particular sections such as the fast Fourier transform computer algorithm, the Gibbs phenomenon, causality, or oscillator wave functions. These are subjects which have not been commonly included under the same cover. Part IV, moreover, may spur his or her interest in new directions. We have tried to give an adequate bibliography whenever our account of an area had to stop for reasons of specialization or space. References are cited by author's name and year of publication, and they are listed alphabetically at the end of the book. A generous number of figures and some tables should enable the reader to browse easily. New literals are defined by "$:=$"; thus $f := A$ means f is defined as the expression A. Vectors and unargumented functions are denoted by lowercase boldface type and matrices by uppercase boldface. Operators appear in "double" type, e.g., $\mathbb{Q}$, $\mathbb{P}$, and sets in script, e.g., $\mathscr{R}$, $\mathscr{C}$. A symbol list is included at the end.

Exercises are used mainly to suggest alternative proofs, extensions to the text material, or cross references, usually providing the answers as well as further comments. They are meant to be read at least cursorily as part of the text. Equations are numbered by chapter.

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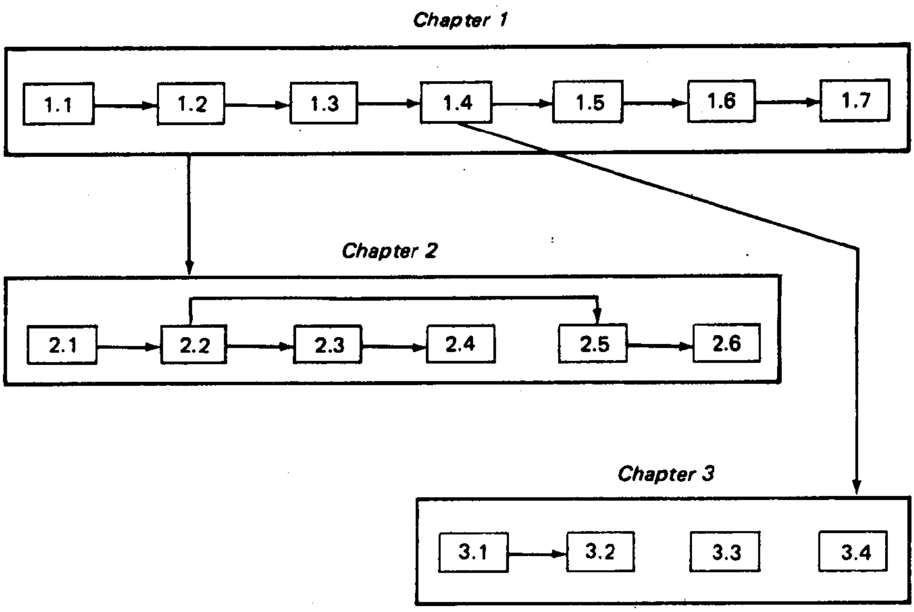
Part I

Finite-Dimensional Vector Spaces and the Fourier Transform

In this part we develop the mathematical framework of finite-dimensional Fourier transforms and give the basics of two fields where it has found fruitful application: in the analysis of coupled systems and in communication theory and technology.

Chapter 1 deals with complex vector analysis in N dimensions and leads rather quickly to the tools of Fourier analysis: unitary transformations and self-adjoint operators. The uncoupling of lattices representing one-dimensional crystals and electric *RLC* networks is undertaken in Chapter 2. We examine in detail the fundamental solutions, normal modes, and traveling waves for first-neighbor interactions in simple crystal lattices and extend these to farther-neighbor, molecular, and diatomic crystals. The Fourier formalism is also used to describe the analytical mechanics of these systems: phase space, energy, evolution operators, and other conservation laws. Chapter 3 introduces convolution and correlation, sketching their use in filtering, windowing, and modulation of signals and their detection in the presence of background noise. The workings of the fast Fourier transform (FFT) computation algorithm are given in Section 3.3. Finally, in Section 3.4, some properties of Fourier series and integral transforms (Parts II and III) are put in the form of corresponding properties of the finite Fourier transform on vector spaces whose dimension grows without bound.

Chapters 2 and 3 are independent of each other and can be chosen according to the reader's interest. With the first choice, Sections 1.6 and 1.7 will be particularly needed. The understanding of Chapter 3, on the other hand, does not require basically more than Sections 1.1–1.4. Before going to the following parts in this text, the reader may find Section 3.4 useful. Table 1.1, which gives the main properties of the finite Fourier transform, is placed at the end of Chapter 1.



I

Concepts from Complex Vector Analysis and the Fourier Transform

In this chapter we present the basic properties of complex vector spaces and the Fourier transform. Sections 1.1 and 1.2 prepare the subject through the standard definitions of linear independence, bases, coordinates, inner product, and norm. In Section 1.3 we introduce linear transformations in vector spaces, emphasizing the conceptual difference between passive and active ones: the former refer to changes in reference coordinates, while the latter imply a "physical" process actually transforming the points of the space. Permutations of reference axes and the Fourier transformation are prime examples of coordinate changes (Section 1.4), while the second-difference operator in particular and self-adjoint operators in general (Section 1.5) will be important in applications. We give, in Section 1.6, the elements of invariance group considerations for a finite N-point lattice. Finally, in Section 1.7 we examine the axes of a transformation and develop the properties of self-adjoint and unitary operators.

If the reader so wishes, he can proceed from Section 1.4 directly to Chapter 3 for applications in communication and the fast Fourier transform algorithm. The rest of the sections are needed, however, for the treatment of coupled systems in Chapter 2.

1.1. N-Dimensional Complex Vector Spaces

The elements of real vector analysis are surely familiar to the reader, so the material in this section will serve mainly to fix notation and to enlarge slightly the concepts of this analysis to the field $\mathscr{C}$ of complex numbers.

1.1.1. Axioms

Let $c_1, c_2, \ldots$ be complex numbers, elements of $\mathscr{C}$, and let $\mathbf{f}_1, \mathbf{f}_2, \ldots$ be the elements of a set $\mathscr{V}$ called *vectors* and denoted by boldface letters. We shall allow for two operations within $\mathscr{V}$:

(a) To every pair $\mathbf{f}_1$ and $\mathbf{f}_2$ in $\mathscr{V}$, there is an associated element $\mathbf{f}_3$ in $\mathscr{V}$, called the *sum* of the pair: $\mathbf{f}_3 = \mathbf{f}_1 + \mathbf{f}_2$.
(b) To every $\mathbf{f} \in \mathscr{V}$ ("$\mathbf{f}$ element of $\mathscr{V}$") and every $c \in \mathscr{C}$, there is an associated element $c\mathbf{f}$ in $\mathscr{V}$, referred to as the product of $\mathbf{f}$ by c.

With respect to the sum, $\mathscr{V}$ must satisfy the following:

(a1) *Commutativity:* $\mathbf{f}_1 + \mathbf{f}_2 = \mathbf{f}_2 + \mathbf{f}_1$,
(a2) *Associativity:* $(\mathbf{f}_1 + \mathbf{f}_2) + \mathbf{f}_3 = \mathbf{f}_1 + (\mathbf{f}_2 + \mathbf{f}_3)$,
(a3) $\mathscr{V}$ must contain a *zero* vector $\mathbf{0}$ such that $\mathbf{f} + \mathbf{0} = \mathbf{f}$ for all $\mathbf{f} \in \mathscr{V}$,
(a4) For every $\mathbf{f} \in \mathscr{V}$, there exists a $(-\mathbf{f}) \in \mathscr{V}$ such that $\mathbf{f} + (-\mathbf{f}) = \mathbf{0}$.

With respect to the product it is required that $\mathscr{V}$ satisfy

(b1) $1 \cdot \mathbf{f} = \mathbf{f}$,
(b2) $c_1(c_2\mathbf{f}) = (c_1c_2)\mathbf{f}$.

Finally, the two operations are to intertwine *distributively*, i.e.,

(c1) $c(\mathbf{f}_1 + \mathbf{f}_2) = c\mathbf{f}_1 + c\mathbf{f}_2$,
(c2) $(c_1 + c_2)\mathbf{f} = c_1\mathbf{f} + c_2\mathbf{f}$.

The last requirement relates the sum in $\mathscr{C}$ with the sum in $\mathscr{V}$. We use the same symbol "+" for both. Immediate consequences of these axioms are $0\mathbf{f} = \mathbf{0}$ and $(-1)\mathbf{f} = -\mathbf{f}$.

1.1.2. Linear Independence

Except for allowing the numbers $c_1, c_2, \ldots$ to be complex, the main concepts from ordinary vector analysis remain unchanged: A set of (nonzero) vectors $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_N$ is said to be *linearly independent* when

$$\sum_{n=1}^{N} c_n\mathbf{f}_n = \mathbf{0} \Leftrightarrow c_n = 0, \qquad n = 1, 2, \ldots, N. \tag{1.1}$$

If the implication to the right does not hold, the set of vectors is said to be linearly *dependent*. A complex vector space $\mathscr{V}$ is said to be *N-dimensional* when it is possible to find at most N linearly independent vectors. We affix N to $\mathscr{V}$ as a superscript: $\mathscr{V}^N$. Let $\{\boldsymbol{\varepsilon}_n\}_{n=1}^N \coloneqq \{\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \ldots, \boldsymbol{\varepsilon}_N\}$ be a maximal set of linearly independent vectors, called a *basis* for $\mathscr{V}^N$. We can then express any $\mathbf{f} \in \mathscr{V}^N$ as a linear combination of the basis vectors as

$$\mathbf{f} = \sum_{n=1}^{N} f_n \boldsymbol{\varepsilon}_n, \tag{1.2}$$

where $f_n \in \mathscr{C}$ is the *nth coordinate* of $\mathbf{f}$ with respect to the basis $\{\boldsymbol{\varepsilon}_n\}_{n=1}^N$. If $\mathbf{f}$ has coordinates $\{f_n\}_{n=1}^N$ and $\mathbf{g}$ coordinates $\{g_n\}_{n=1}^N$, then the coordinates of a vector $\mathbf{h} = a\mathbf{f} + b\mathbf{g}$ will be $h_n = af_n + bg_n$ for $n = 1, 2, \ldots, N$, as implied by (1.1) and the linear independence of the basis vectors. The vector $\mathbf{0}$ has all its coordinates zero.

1.1.3. Canonical Representation

Any two N-dimensional vector spaces are isomorphic, as we need only establish a one-to-one correspondence between the basis vectors. A most convenient realization of $\{\boldsymbol{\varepsilon}_n\}_{n=1}^N$ is given through the *canonical* column-vector representation

$$\boldsymbol{\varepsilon}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{\varepsilon}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \ldots, \boldsymbol{\varepsilon}_N = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad \text{i.e., } \mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{N-1} \\ f_N \end{pmatrix}. \tag{1.3}$$

Throughout Part I, we shall consider finite-dimensional complex vector spaces.

Exercise 1.1. Map the complex vector space $\mathscr{V}^N$ onto a $2N$-dimensional *real* vector space (i.e., only real numbers allowed). You can number the basis vectors in the latter as $\boldsymbol{\varepsilon}_n^R := \boldsymbol{\varepsilon}_n$ and $\boldsymbol{\varepsilon}_{N+n}^I := i\boldsymbol{\varepsilon}_n$, $n = 1, 2, \ldots, N$. (Any other choice?) How do the coordinates of a vector $\mathbf{f} \in \mathscr{V}^N$ relate to the coordinates of the corresponding vector in the real space?

For economy of notation we shall henceforth indicate summations as in (1.2) by $\sum_n$, the range of the index being implied by the context. Double sums will appear as $\sum_{n,m}$, etc. If any ambiguities should arise, we shall revert to the full summation symbol.

1.2. Inner Product and Norm in $\mathscr{V}^N$

In this section we shall generalize the inner (or "scalar") product and norm of ordinary vector analysis to corresponding concepts in complex vector spaces.

1.2.1. Inner Product

To every ordered pair of vectors **f**, **g** in $\mathscr{V}^N$, we associate a complex number (**f**, **g**), their *inner product*. It has the properties of being *linear* in the second argument, i.e.,

$$(\mathbf{f}, c_1\mathbf{g}_1 + c_2\mathbf{g}_2) = c_1(\mathbf{f}, \mathbf{g}_1) + c_2(\mathbf{f}, \mathbf{g}_2), \tag{1.4}$$

and *antilinear* in the first,

$$(c_1\mathbf{f}_1 + c_2\mathbf{f}_2, \mathbf{g}) = c_1^*(\mathbf{f}_1, \mathbf{g}) + c_2^*(\mathbf{f}_2, \mathbf{g}), \tag{1.5}$$

where the asterisk denotes complex conjugation. Such an inner product is thus a *sesquilinear* ("1½ linear") operation: $\mathscr{V}^N \times \mathscr{V}^N \to \mathscr{C}$. We shall assume that the inner product is *positive*; that is, $(\mathbf{f}, \mathbf{f}) > 0$ for every $\mathbf{f} \neq \mathbf{0}$.

1.2.2. Orthonormal Bases

Two vectors whose inner product is zero are said to be *orthogonal*. A basis such that its vectors satisfy

$$(\boldsymbol{\varepsilon}_n, \boldsymbol{\varepsilon}_m) = \delta_{n,m} \coloneqq \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m \end{cases} \tag{1.6}$$

is said to be an *orthonormal basis*. It can easily be shown as in real vector analysis, by the Schmidt construction, that one can always find an orthonormal basis for $\mathscr{V}^N$. Conversely, we can define the inner product by demanding (1.6) for a given basis and then extend the definition through (1.4) and (1.5) to the whole space $\mathscr{V}^N$. For two arbitrary vectors **f** and **g** written in terms of the basis, we have

$$\begin{aligned}
(\mathbf{f}, \mathbf{g}) &= \left(\sum_n f_n \boldsymbol{\varepsilon}_n, \sum_m f_m \boldsymbol{\varepsilon}_m\right) && \text{[from (1.2)]} \\
&= \sum_m g_m \left(\sum_n f_n \boldsymbol{\varepsilon}_n, \boldsymbol{\varepsilon}_m\right) && \text{[from (1.4)]} \\
&= \sum_{n,m} f_n^* g_m (\boldsymbol{\varepsilon}_n, \boldsymbol{\varepsilon}_m) && \text{[from (1.5)]} \\
&= \sum_n f_n^* g_n. && \text{[from (1.6)]}
\end{aligned} \tag{1.7}$$

It is now easy to verify that

$$(\mathbf{f}, \mathbf{f}) \geqslant 0, \qquad (\mathbf{f}, \mathbf{f}) = 0 \Leftrightarrow \mathbf{f} = \mathbf{0}, \tag{1.8}$$

$$(\mathbf{f}, \mathbf{g}) = (\mathbf{g}, \mathbf{f})^*. \tag{1.9}$$

[In fact, Eqs. (1.4), (1.8), and (1.9) are sometimes used to *define* the inner product in a vector space: the two sets of axioms are equivalent whenever an orthonormal basis exists. This is the case for finite N-dimensional spaces but not always when N is infinite. In the latter, the definition (1.4)–(1.8)–(1.9) is used.]

1.2.3. Coordinates

The nth coordinate of a vector $\mathbf{f}$ in the orthonormal basis $\{\boldsymbol{\varepsilon}_n\}_{n=1}^N$ is easily recovered from $\mathbf{f}$ itself through the inner product: Performing the inner product of a fixed $\boldsymbol{\varepsilon}_m$ with Eq. (1.2), we find

$$(\boldsymbol{\varepsilon}_m, \mathbf{f}) = \left(\boldsymbol{\varepsilon}_m, \sum_n f_n \boldsymbol{\varepsilon}_n\right) = \sum_n f_n(\boldsymbol{\varepsilon}_m, \boldsymbol{\varepsilon}_n) = f_m. \tag{1.10}$$

Hence, we can write

$$\mathbf{f} = \sum_n \boldsymbol{\varepsilon}_n(\boldsymbol{\varepsilon}_n, \mathbf{f}). \tag{1.11}$$

1.2.4. Schwartz Inequality

Two vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ were said to be orthogonal if $(\mathbf{f}_1, \mathbf{f}_2) = 0$. On the other hand, two vectors $\mathbf{g}_1$ and $\mathbf{g}_2$ are *parallel* if $\mathbf{g}_1 = c\mathbf{g}_2$, $c \in \mathscr{C}$, in which case

$$(\mathbf{g}_1, \mathbf{g}_2) = c^*(\mathbf{g}_2, \mathbf{g}_2) = c^{-1}(\mathbf{g}_1, \mathbf{g}_1) = [c^*c^{-1}(\mathbf{g}_1, \mathbf{g}_1)(\mathbf{g}_2, \mathbf{g}_2)]^{1/2}, \tag{1.12}$$

where, note, $|c^*c^{-1}| = 1$. For $|(\mathbf{f}, \mathbf{g})|$, zero is a lower bound, while, in the event $\mathbf{f}$ and $\mathbf{g}$ are parallel, $|(\mathbf{f}, \mathbf{g})| = [(\mathbf{f}, \mathbf{f})(\mathbf{g}, \mathbf{g})]^{1/2}$. These are the extreme values, as stated in the well-known *Schwartz inequality*:

$$|(\mathbf{f}, \mathbf{g})|^2 \leqslant (\mathbf{f}, \mathbf{f})(\mathbf{g}, \mathbf{g}). \tag{1.13}$$

We can prove (1.13) as follows. Consider the vector $\mathbf{f} - c\mathbf{g}$. Then, because of (1.8),

$$0 \leqslant (\mathbf{f} - c\mathbf{g}, \mathbf{f} - c\mathbf{g}) = (\mathbf{f}, \mathbf{f}) - c(\mathbf{f}, \mathbf{g}) - c^*(\mathbf{g}, \mathbf{f}) + |c|^2(\mathbf{g}, \mathbf{g}). \tag{1.14}$$

Now choose (for $\mathbf{g} \neq \mathbf{0}$)

$$c = (\mathbf{f}, \mathbf{g})^*/(\mathbf{g}, \mathbf{g}). \tag{1.15}$$

Replacement in (1.14) and a rearrangement of terms yield (1.13).

1.2.5. Norm

The *norm* (or *length*) of a vector $\mathbf{f} \in \mathscr{V}^N$ is defined as

$$\|\mathbf{f}\| := (\mathbf{f}, \mathbf{f})^{1/2}. \tag{1.16}$$

It is a mapping from $\mathscr{V}^N$ onto $\mathscr{R}^+$ (the nonnegative halfline), having the properties

$$\|\mathbf{f}\| \geqslant 0, \qquad \|\mathbf{f}\| = 0 \Leftrightarrow \mathbf{f} = \mathbf{0}, \tag{1.17}$$

$$\|c\mathbf{f}\| = |c|\,\|\mathbf{f}\|, \tag{1.18}$$

$$\|\mathbf{f} + \mathbf{g}\| \leqslant \|\mathbf{f}\| + \|\mathbf{g}\|. \tag{1.19}$$

Equations (1.17) and (1.18) are easily proven from (1.8) and (1.4)–(1.5), while Eq. (1.19) is the *triangle inequality*, which states, quite geometrically, that the length of the sum of two vectors cannot exceed the sum of the lengths of the vectors. It can be proven from (1.14), setting $c = -1$, that

$$\begin{aligned} 0 \leqslant \|\mathbf{f} + \mathbf{g}\|^2 &= \|\mathbf{f}\|^2 + 2\,\mathrm{Re}(\mathbf{f}, \mathbf{g}) + \|\mathbf{g}\|^2 \\ &\leqslant \|\mathbf{f}\|^2 + 2|(\mathbf{f}, \mathbf{g})| + \|\mathbf{g}\|^2 \qquad (\text{from } \mathrm{Re}\, z \leqslant |z|) \\ &\leqslant \|\mathbf{f}\|^2 + 2\|\mathbf{f}\|\cdot\|\mathbf{g}\| + \|\mathbf{g}\|^2 \qquad [\text{from (1.13)}]. \end{aligned} \tag{1.20}$$

The square root of the second and last terms yields Eq. (1.19).

Exercise 1.2. From (1.14) show that

$$\|\mathbf{f} - \mathbf{g}\| \geqslant |\, \|\mathbf{f}\| - \|\mathbf{g}\|\, |. \tag{1.21}$$

This is another form of the triangle inequality.

We have obtained the properties of the norm, Eqs. (1.17)–(1.19), as consequences of the definition and properties of the inner product. The abstract definition of a *norm*, however, is that of a mapping from $\mathscr{V}^N$ onto $\mathscr{R}^+$, with properties (1.17)–(1.19). It is a weaker requirement than that of an inner product and quite independent of it. The definition (1.16) only represents a particular kind of norm. Again, in infinite-dimensional spaces one may define a norm but have no inner product.

Exercise 1.3. Prove the *polarization* identity

$$(\mathbf{f}, \mathbf{g}) = \tfrac{1}{4}(\|\mathbf{f} + \mathbf{g}\|^2 - \|\mathbf{f} - \mathbf{g}\|^2) + i\tfrac{1}{4}(\|\mathbf{f} - i\mathbf{g}\|^2 - \|\mathbf{f} + i\mathbf{g}\|^2). \tag{1.22}$$

Note that this identity hinges on the validity of (1.16). It *cannot* be used to define an inner product from a norm.

Exercise 1.4. Define the complex angle between two vectors by

$$\cos\Theta \coloneqq (\mathbf{f}, \mathbf{g})/\|\mathbf{f}\|\cdot\|\mathbf{g}\|, \qquad \Theta = \theta_R + i\theta_I. \tag{1.23}$$

Show that this restricts Θ to a region $|\sinh\theta_I| \leqslant |\sin\theta_R| \leqslant 1$.