SOUND ABSORBING MATERIALS

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PREFACE

The current acoustical literature provides us with numerous data and long tables concerning the sound absorption coefficient of the various materials that are on the market. These data together with the physical features of the material (the thickness of the layer, the porosity, the diameter of the pores, its stiffness) provide the skilled designer of absorbing materials with some idea as to how new materials can adequately be designed. In this book we have tried to give the design a more scientific basis. The principles underlying the wave propagation through media — porous or not — are described. The wave propagation through porous media has been treated at considerable length, being of great importance for the majority of absorbing materials.

Except for the last chapter (VIII) all theoretical considerations and measurements are confined to normal incidence of sound.

With a view to the increasing application of absorbing materials behind perforated panels, due attention is paid to this subject (chapter VII).

The origin of the book has been a request of the editor to the first author for a book on the subject of acoustical materials. The latter wrote the text in the grim war winter of 1944—1945 when all laboratory work was utterly impossible. Due to a serious shortage of printing facilities the actual printing was delayed so much that it was considered necessary to review the text on account of the publications that had become available since the war. At the first author's request the second author accepted the task to do this. It turned out that this meant hardly less than rewriting the whole text. Moreover the chapter on resonators was not included in the original scheme and it due to the initiative of the second author.

As far as we know a book with a similar aim does not exist, which may serve as a justification of adding a new book to the extensive acoustical literature. Furthermore, the existence of the latter may help to meet certain shortcomings in this book, of which the authors are convinced there are. May it, nevertheless, turn out to be valuable for the scientific manufacturer.

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CHAPTER I

SIMPLE THEORY OF SOUND ABSORPTION BY HOMOGENEOUS LAYERS

§ 1 PROPAGATION CONSTANT AND WAVE IMPEDANCE

If we consider a plane sound wave travelling in the direction of the positive x-axis in a homogeneous isotropic medium extending to infinity, the sound pressure depends upon the time t and on the distance x as a damped sine:

$$p(x) = A \exp \left\{ j\omega\left(t - \frac{x}{c}\right) - \alpha x \right\} \qquad (1.01)$$

in which $j^2 = -1$, $\omega = 2\pi \cdot$ frequency, $c =$ velocity of propagation of sound, and exp (...) is the well-known symbolical notation for $e^{(\ldots)}$. At the site $x = 0$ we have $p(0) = A \exp j\omega t$. Putting $\omega/c = \beta$ and $\alpha + j\beta = \gamma$, we obtain for the damped sine the shorter analytical form

$$p(x) = p(0) \exp(-\gamma x).$$

The constant γ, which, apart from its dependence on ω, is determined by the nature of the medium is called the *propagation constant* of this medium; its real part α is called the *attenuation constant* and its imaginary part β the *phase constant.*

In the same way the velocity of specific volume displacement, v, can be expressed by a similar formula

$$v(x) = v(0) \exp(-\gamma x),$$

in which the same constant γ enters as in the formula for p, because in a travelling wave the ratio p/v must be independent of x. We shall use the symbol v for the amount of material volume passing through a unit surface in unit time, or the volume current density. Only in a homogeneous medium (free air, compact solid medium) is v identical with the material velocity. For example, in a porous medium with rigid solid skeleton, the velocity of volume displacement v is smaller than the velocity of the vibrating air, the ratio being equal to that of the volume of the accessible holes to the total volume of the medium, h. The constant h is called the *porosity* or *cavity factor* and is one of the elementary properties of the material which will play an important rôle in the following pages.

In analogy to electrical practice the quotient

$$z(x) = p(x)/v(x) \tag{1.02}$$

is called the specific acoustic impedance at the site x. For an unlimited medium, z must be independent of x. This impedance is a material constant and is called its *wave impedance*, represented by W. As p and v are generally not in phase with each other, W is a complex quantity.

If a periodical pressure $p(0)$ is applied at the site $x=0$ of an unlimited medium, the dependence of p and v upon x and t is fully determined by the two quantities γ and W. Thus γ and W fully determine the acoustical behaviour of the medium.

§ 2 IMPEDANCE OF A LAYER OF FINITE THICKNESS

We shall next consider a layer of uniform constitution, determined by the properties γ and W, having a thickness l ($x=0$ to $x=l$). Supposing it to be loaded at $x=l$ by the arbitrary complex impedance z_2, what is, then, the impedance z_1 at $x=0$ (Fig. 1)? Part of the sound wave will be reflected at the end $x=l$, so that p is the superposition of an incoming wave and a reflected wave:

$$p(x) = p_i \exp\{\gamma(l-x)\} + p_r \exp\{-\gamma(l-x)\}$$
$$v(x) = (p_i/W)\exp\{\gamma(l-x)\} - (p_r/W)\exp\{-\gamma(l-x)\},$$

in which p_i and p_r are obviously the possibly complex pressure of the incoming and the reflected wave at $x=l$, just inside the layer.

The pressure adds as a scalar, the velocity as a vector; hence the minus sign in the formula for v. As a boundary condition we have $p(l)/v(l) = z_2$. With the aid of this it may easily be verified that

$$p_r/p_i = (z_2 - W)/(z_2 + W), \tag{1.03}$$

and introducing this into the preceding equations one finds for the impedance at the site $x=0$

$$z_1 = W \frac{z_2 \cosh \gamma l + W \sinh \gamma l}{z_2 \sinh \gamma l + W \cosh \gamma l}, \tag{1.04}$$

a formula wellknown from the theory of electric cables and filters.

This formula contains, as a special case, that of the medium extending to infinity. Then $z_2 = W$ and the formula gives $z_1 = z_2 = W$. Another special case is that in which $z_2 = \infty$, in which case

$$z_1 = W \coth \gamma l. \tag{1.05}$$

This latter case is of great importance for the investigations, because this is the one which can easily be realized by loading the sheet of absorbent material with a completely hard back wall. Backing absorbent layers with a rigid wall is a normal way of

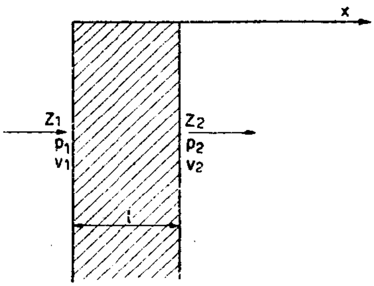

Fig. 1
Layer backed by an impedance z_2

applying these materials in practice, so that this case is at the same time of great practical importance.

The other limiting case is obtained by putting $z_2 = 0$. Then (1.04) yields

$$z_1 = W \tanh \gamma l. \tag{1.06}$$

Because the impedance of a layer of air with a thickness of one quarter of a wave length, backed by a rigid wall, is zero (see (1.05) remembering that $\gamma = j\omega/c$, since there is no damping), this case can be realized by placing the absorbent layer at a distance of $^1/_4$ λ from the rigid wall.

It is now possible to calculate in principle the acoustic impedance of any combination of sheets of different characters and lengths by the successive application of (1.04) (Fig. 2). One starts by computing z_{n-1} from z_n, then z_{n-2} from z_{n-1}, and so on, until finally z_1 is found. The calculation is complicated and tiresome and might better be replaced by a geometrical method [1].

§ 3 WAVE IMPEDANCE W_0 OF FREE AIR

The theoretical deduction of the quantities W and γ for any medium is always accomplished in essentially the same way by starting from the equation of motion for the vibrating medium and from the equation of continuity (conservation of mass). We shall now consider the simplest case, viz., that of free air without taking into account the effects of damping.

The equation of motion is found by applying Newton's

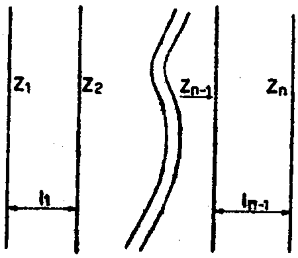

Fig. 2

Successive computation of the impedance of a multilayer system

equation (force = mass × acceleration) to a thin layer of air of thickness dx:

$$-\frac{\partial p}{\partial x} = \rho_0 \frac{\partial v}{\partial t}, \qquad (1.07)$$

in which ρ_0 = density. The equation of continuity is

$$-\frac{\partial v}{\partial x} = \frac{1}{\rho_0}\frac{\partial \rho}{\partial t} = \frac{1}{K_0}\frac{\partial p}{\partial t}, \qquad (1.08)$$

in which

$$K_0 = \frac{dp}{d\rho/\rho_0}.$$

Eliminating v by differentiating (1.07) with respect to x

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[1] See e.g., Feldtkeller, Vierpoltheorie and Chapter VI, § 7, Fig. 67.

and (1.08) with respect to t and equating the expressions for $\partial^2 v/\partial x\,\partial t$ thus obtained, gives a differential equation for p

$$\frac{\partial^2 p}{\partial x^2} = \frac{\rho_0}{K_0}\,\frac{\partial^2 p}{\partial t^2}. \tag{1.09}$$

Since we expect a solution of the form $p = A \exp(j\omega t) \exp(-\gamma_0 x)$ we may put: $-\partial/\partial x = \gamma_0$ and $\partial/\partial t = j\omega$, so that (1.09) reduces to

$$\gamma_0{}^2 = -\frac{\rho_0}{K_0}\,\omega^2,$$

from which we get

$$\gamma_0 = \pm j\omega \sqrt{\rho_0/K_0}. \tag{1.10}$$

By comparing (1.10) with (1.01) we see that the physical interpretation of the constant $\sqrt{K_0/\rho_0}$ is the velocity of propagation c_0 of sound waves in free air, the plus (minus) sign for γ_0 having to be taken for waves travelling in (opposite to) the direction of the positive x-axis. Substituting this value of $-\partial/\partial x$ together with $j\omega$ for $\partial/\partial t$ in either equation (1.07) or (1.08) yields for the wave impedance

$$p/v = W_0 = \sqrt{K_0\rho_0} = \rho_0 c_0. \tag{1.11}$$

From (1.11) we learn that the wave impedance is real. By inserting the known values of ρ_0 and c_0 at room temperature it appears that W_0 has the approximate value of 420 kg m^{-2} sec^{-1} (42 cgs).

If plane waves from the air impinge upon a wall of specific impedance z, reflection will take place; the ratio of the pressure of the reflected wave to that of the incoming wave just before the reflecting boundary is given by (1.03), which runs in the present notation

$$r = \frac{p_r}{p_i} = \frac{z - W_0}{z + W_0}. \tag{1.12}$$

We shall call this ratio the complex reflection coefficient. By squaring the absolute value one obtains the reflection coefficient for the energy, which is complementary to the absorption coefficient:

$$a_0 = 1 - \left|\frac{p_r}{p_i}\right|^2 = 1 - \left|\frac{z - W_0}{z + W_0}\right|^2, \tag{1.13}$$

the index zero being added because this coefficient applies only to normal incidence.

What is said of free air in this section will be shown to be of quite general validity. The equations of continuity and motion may in many cases be written down in exactly the same form (1.07 and 1.08), although the constants ρ and K will have a different meaning; they will be dependent on several more or less elementary properties of the medium and, finally, will have to be considered as complex quantities.

In order to avoid confusion, all symbols relating to free air will, when necessary, be provided with an index zero, as was done in this section: ρ_0, K_0, γ_0, W_0, c_0.

A very important way of application of absorbing materials is the direct fixing of a layer of such material to a rigid back wall. In this case the impedance at the front side was shown to be (c.f. § 2)

$$z = W \coth \gamma l. \tag{1.05}$$

Substituting the general expressions for W and γ of (1.10) and (1.11) yields

$$z = \sqrt{K\rho} \quad \coth \quad j\omega l \sqrt{\rho/K} \tag{1.14}$$

as a general expression for the impedance of any layer backed by a rigid wall. We have to make an exception for some porous flexible layers. The question of the computation of K and ρ will be considered in § § 7 to II § 6.

§ 4 GEOMETRY IN THE COMPLEX PLANE

The understanding of such formulae as (1.05) and (1.11) may be facilitated by plotting the function coth γl and other complex functions z of ω in the complex plane. If we allow ω to assume all values from 0 to ∞ the function under consideration follows a certain contour in the complex plane. This curve, as a rule, appeals more directly to our imagination than does an analytical formula. The point is to separate the real from the imaginary part so that z takes the form $z(\omega) = x(\omega) + jy(\omega)$, and to plot z in the plane with x and y (belonging to the same value of the parameter ω) as coordinates.

For example, the function $z = 1 - j\omega$ is a straight line starting at the point $x = 1$ of the real axis and going downwards (Fig. 3); $z = \exp(j\omega) = \cos\omega + j\sin\omega$ is the circle with unit radius.

Multiplying a complex number by $\exp(j\omega)$ means turning it

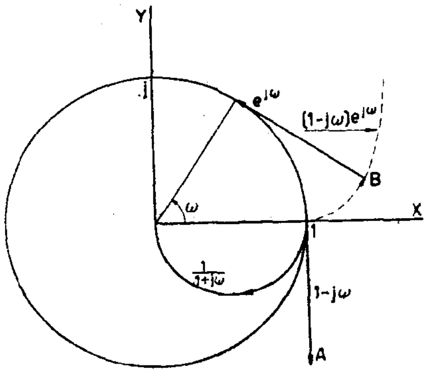

Fig. 3
Complex representation of the circle involute

round the origin through an angle ω. So by multiplying the point A (Fig. 3) by $\exp(j\omega)$ we come to the point B and it is obvious,

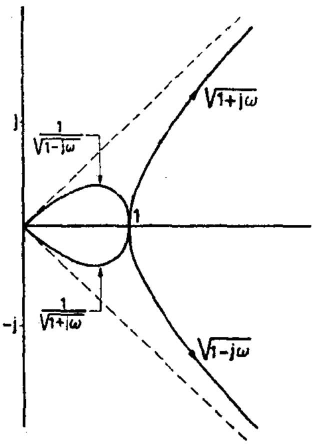

Fig. 4
Complex representation of a hyperbola and a lemniscate

that B lies on the involute of the circle; hence $z = (1 - j\omega)\exp(j\omega)$ is the circle involute.

The function $z = \sqrt{1 + j\omega}$ is represented by a rectangular hyperbola (Fig. 4), which may be proved as follows. Put $z = x +$

$jy = \sqrt{1 + j\omega}$ and square: $x^2 - y^2 + 2\,j\,xy = 1 + j\omega$. Separating the real from the imaginary parts shows us that $x^2 - y^2 = 1$ and this is the common formula for the rectangular hyperbola. Further $2\,xy = \omega$; now since ω is always positive x and y have the same sign. The function $\sqrt{1 + j\omega}$ is represented by the upper branch of the hyperbola, the lower branch being represented by $\sqrt{1 - j\omega}$. The generalization $z = \sqrt{1 \pm jf(\omega)}$, where $f(\omega)$ is an arbitrary real function of ω, leads to the same hyperbola but the ω-scale is changed. This remark is one of a general validity, we may always pass on to a new ω-scale on the same curve by substituting $f(\omega)$ for ω.

Inverting a complex number $z = M \exp(j\varphi)$ gives $\frac{1}{z} = \frac{1}{M} \exp(-j\varphi)$, so the modulus is inverted, and the sign of the argument is the opposite of the original one. As the lemniscate is the inverse curve of the hyperbola the former is represented by the functions $(1 \pm j\omega)^{-\frac{1}{2}}$.

By inverting a straight line, one obtains, as is well known from elementary geometry, a circle. In Fig. 3 the inversion of the half line $1 + j\omega$ is drawn as a half circle running from the point $x = 1$ on the real axis towards the origin.

More examples illustrating the simple geometrical representation of functions by contours and vice versa may easily be obtained.

§ 5 GEOMETRICAL REPRESENTATION OF COTH γl

In Fig. 5 the function $\coth \gamma l$ is plotted with the aid of[1] the formula

$$\coth \gamma l = \coth(\alpha + j\beta) l = \frac{\sinh 2\,\alpha l - j \sin 2\,\beta l}{\cosh 2\,\alpha l - \cos 2\,\beta l} \qquad (1.15)$$

for constant values of α and β. It looks like a logarithmic spiral. And indeed, for large values of the argument, $\coth \gamma l$ *is approximated* by a logarithmic spiral. We arrive at this result, because for large values of γl, $\coth \gamma l \sim 1 + 2 \exp(-2\,\gamma l)$.

Leaving aside for the moment the shift over the distance 1 in the direction of the real axis, the function

[1] J. Rybner, Nomograms of Complex Hyperbolic Functions, p. 25, Copenhagen 1947.

$$2 \exp(-2\gamma l) = 2 \exp(-2\alpha l) \cdot \exp(-2 j\beta l)$$

represents a spiral with modulus $M = 2 \exp(-2\alpha l)$ and argument $\varphi = -2\beta l$. Hence the relation between M and φ is

$$M = 2 \exp\left(\frac{\alpha}{\beta}\varphi\right),$$

and this is, for a constant value of α/β, the usual formula for

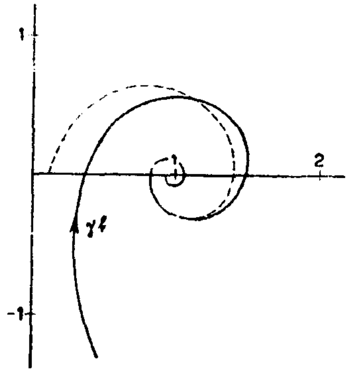

Fig. 5
Complex representation of coth γl

the logarithmic spiral in polar coordinates. It is indicated by a dashed curve in Fig. 5. In the same way it can be shown that tanh γl converges towards the logarithmic spiral $1 - 2 \exp(-2\gamma l)$.

The geometrical interpretation of the constant α/β is this,

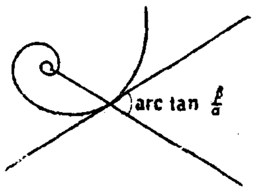

Fig. 6
The slope of coth γl

that it is the tangent of the "slope" of the spiral, which is constant along the logarithmic spiral (Fig. 6). The greater α, the greater the slope, and the more rapidly the function converges towards its apex. Figs. 7 and 8 represent two cases with slopes 1 and 0.1 respectively.

For an arbitrary medium α and β are both functions of ω and so is the slope α/β. Taking l as a constant, γl varies because of the variations in α and β with ω. In this general case we get a

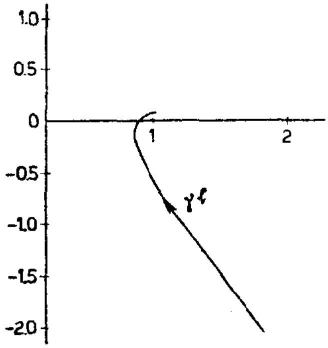

Fig. 7
A coth with slope 1,
strong damping

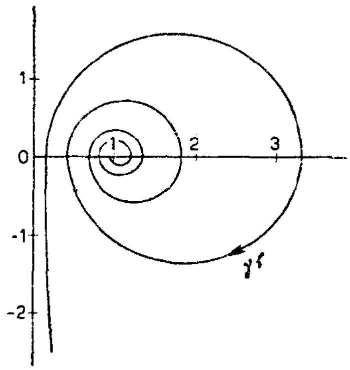

Fig. 8
A coth with slope 0.1,
weak damping

modification of the logarithmic spiral with a slope varying along the curve. In Fig. 9 the case $\alpha/\beta = \text{const.}/\omega$ is illustrated; the

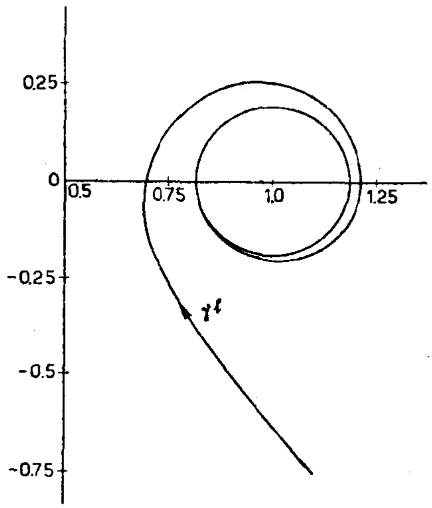

Fig. 9
A coth with a slope inversely proportional with frequency

slope approaches the value of zero for large values of ω, the contour approaches to an asymptotic circle.

Multiplication of $\coth \gamma l$ by a complex number $W = M \exp j\varphi$ (cf. 1.05) means enlarging the contour by a factor M and rotating it through an angle φ around the origin, as in the example of the circle involute. Multiplication by a complex number W, which in itself is a function of ω, is accomplished by applying this process for each point of the spiral, taking together corresponding points (for the same value of ω) of the multiplier W and $\coth \gamma l$. If W does not run fast through the diagram, something resembling a spiral may still be seen; if, on the other hand, W runs fast, the spiral appearance may be lost, and a kind of damped sine curve with curved axis may result, this curved axis being the contour of W.

§ 6 GEOMETRICAL REPRESENTATION OF FORMULA (1.12)

We shall now give a geometrical interpretation of (1.12)

$$r = \frac{z - W_0}{z + W_0}. \tag{1.12}$$

We plot z in a complex plane and also plot the points W_0 and $-W_0$ on the real axis. $z - W_0$ is the vector going from W_0 to z; $z + W_0$ is the vector going from $-W_0$ to z, and the complex reflection coefficient is the quotient of these two vectors. Now in general the quotient of two vectors

$$\frac{M_1 \exp (j\varphi_1)}{M_2 \exp (j\varphi_2)} = \frac{M_1}{M_2} \exp \{ j(\varphi_1 - \varphi_2) \}$$

is a vector with an absolute value equal to the ratio of the *absolute* values of the two original vectors and an argument equal to the differences of the arguments.

Hence the absolute value of p_r/p_i is the ratio of the two vectors, shown in Fig. 10, its argument, i. e., the phase angle by which p_r is ahead of p_i, is the angle Δ in fig. 10[1].

The absolute value of r and therefore, the value of the absorption coefficient $a_0 = 1 - |r|^2$ remains constant along contours for which $M_1/M_2 =$ constant and from elementary geometry we know that these contours are circles. For 100 % absorption $M_1 = 0$, and the circle reduces to the point W_0 (Fig. 11). On the other hand

[1] In accordance with usage in electrotechnics we denote the real part of z by R, its imaginary part by X, Y being reserved for the admittance $Y = 1/z$.

we also know from elementary geometry that the locus of point z with the same value of Δ is also a circle, now going through the point W_0 and $-W_0$ (Fig. 11). Moreover the two sets of circles

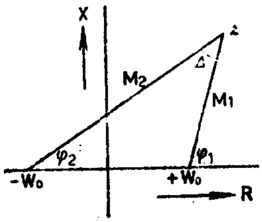

Fig. 10
Modulus and argument of the complex reflection coefficient r (equation 1.12) in the z-plane

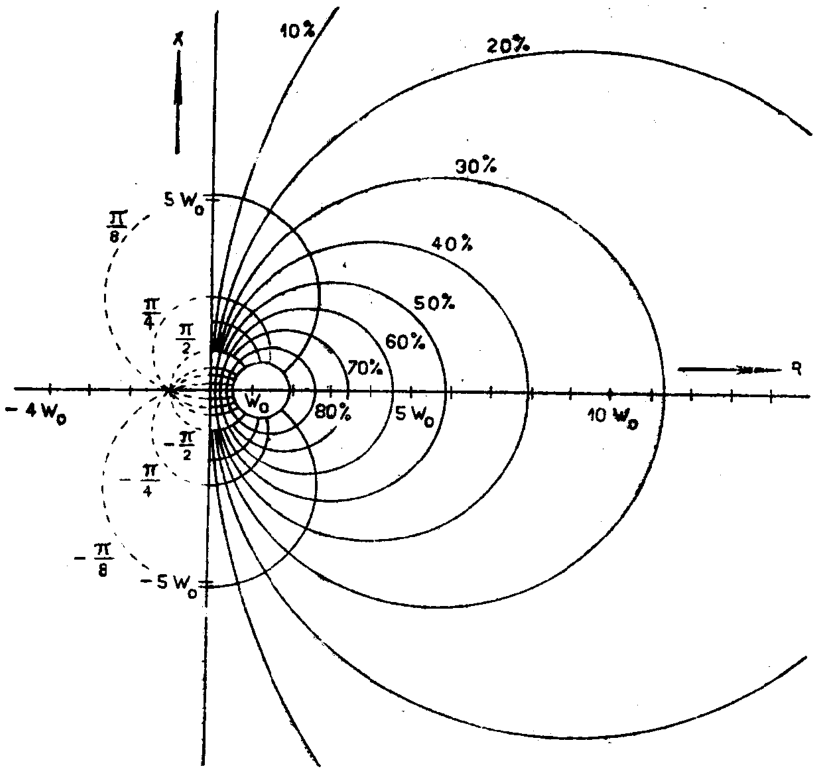

Fig. 11
The complex reflection coefficient r in the impedance plane, phase jump Δ and absorption coefficient a_0, the circle diagram