Graduate Texts in Mathematics

Serge Lang

$SL_2(\mathbf{R})$

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Foreword

Starting with Bargmann's paper on the infinite dimensional representations of $SL_2(\mathbf{R})$, the theory of representations of semisimple Lie groups has evolved to a rather extensive production. Some of the main contributors have been: Gelfand–Naimark and Harish-Chandra, who considered the Lorentz group in the late forties; Gelfand–Naimark, who dealt with the classical complex groups, while Harish-Chandra worked out the general real case, especially through the derived representation of the Lie algebra, establishing the Plancherel formula (Gelfand–Graev also contributed to the real case); Cartan, Gelfand–Naimark, Godement, Harish-Chandra, who developed the theory of spherical functions (Godement gave several Bourbaki seminar reports giving proofs for a number of spectral results not accessible otherwise); Selberg, who took the group modulo a discrete subgroup and obtained the trace formula; Gelfand, Fomin, Pjateckii-Shapiro, and Harish-Chandra, who established connections with automorphic forms; Jacquet–Langlands, who pushed through the connection with L-series and Hecke theory. This history is so involved and so extensive that I am incompetent to give a really good account, and I refer the reader to bibliographies in the books by Warner, Gelfand–Graev–Pjateckii-Shapiro, and Helgason for further information. A few more historical comments will be made in the appropriate places in the book.

It is not easy to get into representation theory, especially for someone interested in number theory, for a number of reasons. First, the general theorems on higher dimensional groups require massive doses of Lie theory. Second, one needs a good background in standard and not so standard analysis on a fairly broad scale. Third, the experts have been writing for each other for so long that the literature is somewhat labyrinthine.

I got interested because of the obvious connections with number theory, principally through Langlands' conjecture relating representation theory to elliptic curves [La 2]. This is a global conjecture, in the adelic theory. I realized soon enough that it was best to acquire a good understanding of the real theory before getting everything on the adeles. I think most people who have worked in representations have looked at $SL_2(\mathbf{R})$ first, and I know this is the case for both Harish and Langlands.

Therefore, as I learned the theory myself it seemed a good idea to write up $SL_2(\mathbf{R})$. The topics are as follows:

1. We first show how a representation decomposes over the maximal compact subgroup K consisting of all matrices

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

and see that an irreducible representation decomposes in such a way that each character of K (indexed by an integer) occurs at most once.

2. We describe the Iwasawa decomposition $G = ANK$, from which most of the structure and theorems on G follow. In particular, we obtain representations of G induced by characters of A.

3. We discuss in detail the case when the trivial representation of K occurs. This is the theory of spherical functions. We need only Haar measure for this, thereby making it much more accessible than in other presentations using Lie theory, structure theory, and differential equations.

4. We describe a continuous series of representations, the induced ones, some of which are unitary.

5. We discuss the derived representation on the Lie algebra, getting into the infinitesimal theory, and proving the uniqueness of any possible unitarization. We also characterize the cases when a unitarization is possible, thereby obtaining the classification of Bargmann. Although not needed for the Plancherel formula, it is satisfying to know that any unitary irreducible representation is infinitesimally isomorphic to a subrepresentation of an induced one from a quasicharacter of the diagonal group. The derived representation of the Lie algebra on the algebraic space of K-finite vectors plays a crucial role, essentially algebraicizing the situation.

6. The various representations are related by the Plancherel inversion formula by Harish-Chandra's method of integrating over conjugacy classes.

7. We give a method of Harish-Chandra to unitarize the "discrete series," i.e. those representations admitting a highest and lowest weight vector in the space of K-finite vectors.

8. We discuss the structure of the algebra of differential operators, with special cases of Harish-Chandra's results on $SL_2(\mathbf{R})$ giving the center of the universal enveloping algebra and the commutator of K. At this point, we have enough information on differential equations to get the one fact about spherical functions which we could not prove before, namely that there are no other examples besides those exhibited in Chapter IV.

The above topics in a sense conclude a first part of the book. The second part deals with the case when we take the group modulo a discrete subgroup. The classical case is $SL_2(\mathbf{Z})$. This leads to inversion formulas and spectral decomposition theorems on $L^2(\Gamma \backslash G)$, which constitute the remaining chapters.

I had originally intended to include the Selberg trace formula over the reals, but in the case of non-compact quotient this addition would have been sizable, and the book was already getting big. I therefore decided to omit it, hoping to return to the matter at a later date.

A good portion of the first part of the book depends only on playing with Haar measure and the Iwasawa decomposition, without infinitesimal considerations. Even when we use these, we are able to carry out the Plancherel formula and the discussion of the various representations without caring whether we have "all" irreducible unitary representations, or "all" spherical functions (although we prove incidentally that we do). A separate chapter deals with those theorems directly involving partial differential equations via the Casimir operator, and analytical considerations using the regularity theorem for elliptic differential equations. The organization of the book is therefore designed for maximal flexibility and minimal *a priori* knowledge. The methods used and the notation are carefully chosen to suggest the approach which works in the higher dimensional case.

Since I address this book to those who, like me before I wrote it, don't know anything, I have made considerable efforts to keep it self-contained. I reproduce the proofs of a lot of facts from advanced calculus, and also several appendices on various parts of analysis (spectral theorem for bounded and unbounded hermitian operators, elliptic differential equations, etc.) for the convenience of the reader. These and my *Real Analysis* form a *sufficient* background.

The Faddeev paper on the spectral decomposition of the Laplace operator on the upper half-plane is an exceedingly good introduction to analysis, placing the latter in a nice geometric framework. Any good senior undergraduate or first year graduate student should be able to read most of it, and I have reproduced it (with the addition of many details left out to more expert readers by Faddeev) as Chapter XIV. Faddeev's method comes from perturbation theory and scattering theory, and as such is interesting for its own sake, as well as to analysts who may know the analytic part and may want to see how it applies in the group theoretic context. Kubota's recent book on Eisenstein series (which appeared while the present book was in production) uses a different method (Selberg–Langlands), and assumes most of the details of functional analysis as known. Therefore, neither Kubota's book nor mine makes the other unnecessary.

It would have been incoherent to expand the present book to a global context with adeles. I hope nevertheless that the reader will be well prepared to move in that direction after having gotten acquainted with $SL_2(\mathbf{R})$. The book by Gelfand–Graev–Pjateckii-Shapiro is quite useful in that respect.

I have profited from discussions with many people during the last two years, some of them at the Williamstown conference on representation theory in 1972. Among them I wish to thank specifically Godement, Harish-Chandra, Helgason, Labesse, Lachaud, Langlands, C. Moore, Sally, Wilfried Schmid, Stein. Peter Lax and Ralph Phillips were of great help in teaching me some PDE. I also thank those who went through the class at Yale and made helpful contributions during the time this book was evolving. I am especially grateful to R. Bruggeman for his careful reading of the manuscript. I also want to thank Joe Repka for helping me with the proofreading.

New Haven, Connecticut Serge Lang
September 1974

Notation

To denote the fact that a function is bounded, we write $f = O(1)$. If f, g are two functions on a space X and $g \geqslant 0$, we write $f = O(g)$ if there exists a constant C such that $|f(x)| \leqslant Cg(x)$ for all $x \in X$. If $X = \mathbf{R}$ is the real line, say, the above relation may hold for x sufficiently large, say $x > x_0$, and then we express this by writing $x \to \infty$. Instead of $f = O(g)$, we also use the Vinogradov notation,

$$f \ll g.$$

On a topological space X, $C(X)$ is the space of continuous functions. If X is a C^∞ manifold (nothing worse than open subsets of euclidean space, or something like $SL_2(\mathbf{R})$, with obvious coordinates, will occur), we let $C^\infty(X)$ be the space of C^∞ functions. We put a lower index c to indicate compact support. Hence $C_c(X)$ and $C_c^\infty(X)$ are the spaces of continuous and C^∞ functions with compact support, respectively.

By the way, $SL_2(\mathbf{R})$ is the group of 2×2 real matrices with determinant 1.

An isomorphism is a morphism (in a category) having an inverse in this category. An automorphism is an isomorphism of an object with itself. For instance, a continuous linear automorphism of a normed vector space H is a continuous linear map $A: H \to H$ for which there exists a continuous linear map $B: H \to H$ such that $AB = BA = I$. A C^∞ isomorphism is a C^∞ mapping having a C^∞ inverse.

If H is a Banach space, we let $\mathrm{En}(H)$ denote the Banach space of continuous linear maps of H into itself. If H is a Hilbert space, we let $\mathrm{Aut}(H)$ be the group of **unitary** automorphisms of H. We let $GL(H)$ be the group of continuous linear automorphisms of H with itself.

If G' is a subgroup of a group G we let

$$G' \backslash G$$

be the space of right cosets of G'. If Γ operates on a set $\mathfrak{H}$, we let

$$\Gamma\backslash\mathfrak{H}$$

be the space of Γ-orbits. Certain right wingers put their discrete subgroup Γ on the right. Gelfand–Graev–Pjateckii-Shapiro and Langlands put it on the left. I agree with the latter, and hope to turn the right wingers into left wingers.

For the convenience of the reader we also include a summary of objects used frequently throughout the book, with a very brief indication of their respective definitions at the end of the book for quick reference.

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Chapter XI The Weil Representation

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Chapter XIII The Continuous Part of $L^2(\Gamma\backslash G)$

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I General Results

§1. THE REPRESENTATION ON $C_c(G)$

Let G be a locally compact group, always assumed Hausdorff. Let H be a Banach space (which in most of our applications will be a Hilbert space). A **representation** of G in H is a homomorphism

$$\pi: G \longrightarrow GL(H)$$

of G into the group of continuous linear automorphisms of H, such that for each vector $v \in H$ the map of G into H given by

$$x \longmapsto \pi(x)v$$

is continuous. One may say that the homomorphism is **strongly continuous**, the strong topology being the norm topology on the Banach space. [We recall here that the **weak topology** on H is that topology having the smallest family of open sets for which all functionals on H are continuous.]

A representation is called **bounded** if there exists a number $C > 0$ such that $|\pi(x)| < C$ for all $x \in G$. If H is a Hilbert space and $\pi(x)$ is unitary for all $x \in G$, i.e. preserves the norm, then the representation π is called **unitary**, and is obviously bounded by 1.

For a representation, it suffices to verify the continuity condition above on a dense subset of vectors; in other words:

Let $\pi: G \longrightarrow GL(H)$ be a homomorphism and assume that for a dense set of $v \in H$ the map $x \longmapsto \pi(x)v$ is continuous. Assume that the image of some neighborhood of the unit element e in G under π is bounded in $GL(H)$. Then π is a representation.

This is trivially proved by three epsilons. Indeed, it suffices to verify the continuity at the unit element. Let $v \in H$ and select v_1 close to v such that

$x \mapsto \pi(x)v_1$ is continuous. We then use the triangle inequality

$$|\pi(x)v - v| \leq |\pi(x)v - \pi(x)v_1| + |\pi(x)v_1 - v_1| + |v_1 - v|$$

to prove our assertion.

A representation $\pi\colon G \to GL(H)$ is locally bounded, i.e. given a compact subset K of G, the set $\pi(K)$ is bounded in $GL(H)$.

Proof. Let K be a compact subset of G. For each $v \in H$ the set $\pi(K)v$ is compact, whence bounded. By the uniform boundedness theorem (*Real Analysis*, VIII, §3) it follows that $\pi(K)$ is bounded in $GL(H)$.

For the convenience of the reader, we recall briefly the uniform boundedness theorem.

Let $\{T_i\}_{i \in I}$ be a family of bounded operators in a Banach space E, and assume that for each $v \in E$ the set $\{T_i v\}_{i \in I}$ is bounded. Then the family $\{T_i\}_{i \in I}$ is bounded, as a subset of $\mathrm{End}(E)$.

Proof. Let C_n be the set of elements $v \in E$ such that

$$|T_i v| \leq n, \qquad \text{all } i \in I.$$

Then C_n is closed, and E is the union of the sets C_n. It follows by Baire's theorem that some C_n contains an open ball. Translating this open ball to the origin yields an open ball B such that the union of the sets $T_i(B)$, $i \in I$, is bounded, whence the family $\{T_i\}_{i \in I}$ is bounded, as desired.

We let $C_c(G)$ denote the space of continuous functions on G with compact support. It is an algebra under convolution, i.e. the product is defined by

$$\varphi * \psi(x) = \int_G \varphi(xy^{-1})\psi(y)\, dy,$$

where dy is a Haar measure on G. We shall assume throughout that G is **unimodular**, meaning that left Haar measure is equal to right Haar measure. For any function f on G we denote by f^- the function $f^-(x) = f(x^{-1})$. Then

$$\int f(x)\, dx = \int f(x^{-1})\, dx = \int f^-(x)\, dx.$$

Remark. When G is not unimodular, then by uniqueness of Haar measure, there is a modular function $\Delta\colon G \to \mathbf{R}^+$ which is a continuous homomorphism into the positive reals, such that

$$\int_G f(xa)\,dx = \Delta(a)\int_G f(x)\,dx.$$

One then has

$$\int_G f(x^{-1})\Delta(x)\,dx = \int_G f(x)\,dx$$

by an obvious argument. It follows that $\Delta(x)\,dx$ is right Haar measure. The typical non-unimodular group which will concern us, but not until Chapter III, is the group of triangular matrices

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$$

For this chapter, you can forget about the non-unimodular case.

The modular function occurs in a slightly more general context than above. Let $\tau: G \to G$ be either an automorphism (group and topological) of G, or an anti-automorphism, meaning

$$(xy)^\tau = y^\tau x^\tau.$$

We write either x^τ or ${}^\tau x$ for the effect of τ on an element $x \in G$. By the invariance of Haar measure, there exists a positive number $\Delta(\tau)$ such that

$$\int_G f(x^\tau)\,dx = \Delta(\tau)\int_G f(x)\,dx,$$

because the expression on the left is a non-trivial invariant positive functional on $C_c(G)$. We have the obvious composition rule

$$\Delta(\tau\sigma) = \Delta(\tau)\,\Delta(\sigma).$$

In many applications, we have $\tau^2 = Id$, and therefore $\Delta(\tau) = 1$, i.e. τ is unimodular. This occurs in the context of matrices, when for instance τ is the transpose.

The basic example of a unimodular group is the group of matrices

$$G = GL_n(\mathbf{R}).$$

The change of variables formula shows that Haar measure on G is equal to

$$\frac{d^+x}{|\det x|^n}$$