Serge Lang

A First Course in Calculus

Fifth Edition

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Foreword

The purpose of a first course in calculus is to teach the student the basic notions of derivative and integral, and the basic techniques and applications which accompany them. The very talented students, with an obvious aptitude for mathematics, will rapidly require a course in functions of one real variable, more or less as it is understood by professional mathematicians. This book is not primarily addressed to them (although I hope they will be able to acquire from it a good introduction at an early age).

I have not written this course in the style I would use for an advanced monograph, on sophisticated topics. One writes an advanced monograph for oneself, because one wants to give permanent form to one's vision of some beautiful part of mathematics, not otherwise accessible, somewhat in the manner of a composer setting down his symphony in musical notation.

This book is written for the students to give them an immediate, and pleasant, access to the subject. I hope that I have struck a proper compromise, between dwelling too much on special details and not giving enough technical exercises, necessary to acquire the desired familiarity with the subject. In any case, certain routine habits of sophisticated mathematicians are unsuitable for a first course.

Rigor. This does not mean that so-called rigor has to be abandoned. The logical development of the mathematics of this course from the most basic axioms proceeds through the following stages:

Set theory	Numbers (i.e. real numbers)
Integers (whole numbers)	Limits
Rational numbers (fractions)	Derivatives and forward.

No one in his right mind suggests that one should begin a course with set theory. It happens that the most satisfactory place to jump into the subject is between limits and derivatives. In other words, any student is ready to accept as intuitively obvious the notions of numbers and limits and their basic properties. Experience shows that the students do *not* have the proper psychological background to accept a theoretical study of limits, and resist it tremendously.

In fact, it turns out that one can have the best of both ideas. The arguments which show how the properties of limits can be reduced to those of numbers form a self-contained whole. Logically, it belongs *before* the subject matter of our course. Nevertheless, we have inserted it as an appendix. If some students feel the need for it, they need but read it and visualize it as Chapter 0. In that case, everything that follows is as rigorous as any mathematician would wish it (so far as objects which receive an analytic definition are concerned). Not one word need be changed in any proof. I hope this takes care once and for all of possible controversies concerning so-called rigor.

Most students will not feel any need for it. My opinion is that epsilon-delta should be entirely left out of ordinary calculus classes.

Language and logic. It is not generally recognized that some of the major difficulties in teaching mathematics are analogous to those in teaching a foreign language. (The secondary schools are responsible for this. Proper training in the secondary schools could entirely eliminate this difficulty.) Consequently, I have made great efforts to carry the student verbally, so to say, in using proper mathematical language. It seems to me essential that students be required to write their mathematics papers in full and coherent sentences. A large portion of their difficulties with mathematics stems from their slapping down mathematical symbols and formulas isolated from a meaningful sentence and appropriate quantifiers. Papers should also be required to be neat and legible. They should not look as if a stoned fly had just crawled out of an inkwell. Insisting on reasonable standards of expression will result in drastic improvements of mathematical performance. The systematic use of words like "let," "there exists," "for all," "if ... then," "therefore" should be taught, as in sentences like:

Let $f(x)$ be the function such that ....
There exists a number such that ....
For all numbers x with $0 < x < 1$, we have ....
If f is a differentiable function and K a constant such that $f'(x) = Kf(x)$, then $f(x) = Ce^{Kx}$ for some constant C.

Plugging in. I believe that it is unsound to view "theory" as adversary to applications or "computations." The present book treats both as complementary to each other. Almost always a theorem gives a tool for more efficient computations (e.g. Taylor's formula, for computing values of functions). Different classes will of course put different emphasis on them, omitting some proofs, but I have found that if no excessive pedantry is introduced, students are willing, and even eager, to understand the reasons for the truth of a result, i.e. its proof.

It is a disservice to students to teach calculus (or other mathematics, for that matter) in an exclusive framework of "plugging in" ready-made formulas. Proper teaching consists in making the student adept at handling a large number of techniques in a routine manner (in particular, knowing how to plug in), but it also consists in training students in knowing some general principles which will allow them to deal with new situations for which there are no known formulas to plug in.

It is impossible in one semester, or one year, to find the time to deal with all desirable applications (economics, statistics, biology, chemistry, physics, etc.). On the other hand, covering the proper balance between selected applications and selected general principles will equip students to deal with other applications or situations by themselves.

Worked-out problems and exercises. For the convenience of both students and instructors, a large number of worked-out problems has been added in the present edition. Many of these have been put in the answer section, to be referred to as needed. I did this for at least two reasons. First, in the text, they might obscure the main ideas of the course. Second, it is a good idea to make students think about a problem before they see it worked out. They are then much more receptive, and will retain the methods better for having encountered the difficulties (whatever they are, depending on individual students) by themselves. Both the inclusion of worked-out examples and their placement in the answer section was requested by students. Unfortunately, the requirements for good teaching, testing, and academic pressures are in conflict here. The *de facto* tendency is for students to object to being asked to think (even if they fail), because they are afraid of being penalized with bad grades for homework. Instructors may either make too strong requirements on students, or may take the path of least resistance and never require anything beyond plugging in new numbers in a type of exercise which has already been worked out (in class or in the book). I believe that testing conditions (limited time, pressures of other courses and examinations) make it difficult (if not unreasonable) to *test* students other than with basic, routine problems. I do not conclude that the course should consist only of this type of material. Some students often take the attitude that if something is not on tests, then why should it be covered in the course? I object very much to this attitude. I have no global solution to these conflicting pressures.

General organization. I have made no great innovations in the exposition of calculus. Since the subject was discovered some 300 years ago, such innovations were out of the question.

I have cut down the amount of analytic geometry to what is both necessary and sufficient for a general first course in this type of mathematics. For some applications, more is required, but these applications are fairly specialized. For instance, if one needs the special properties concerning the focus of a parabola in a course on optics, then that is the place to present them, not in a general course which is to serve mathematicians, physicists, chemists, biologists, and engineers, to mention but a few. I regard the tremendous emphasis on the analytic geometry of conics which has been the fashion for many years as an unfortunate historical accident. What is important is that the basic idea of representing a graph by a figure in the plane should be thoroughly understood, together with basic examples. The more abstruse properties of ellipses, parabolas, and hyperbolas should be skipped.

Differentiation and the elementary functions are covered first. Integration is covered second. Each makes up a coherent whole. For instance, in the part on differentiation, rate problems occur three times, illustrating the same general principle but in the contexts of several elementary functions (polynomials at first, then trigonometric functions, then inverse functions). This repetition at brief intervals is pedagogically sound, and contributes to the coherence of the subject. It is also natural to slide from integration into Taylor's formula, proved with remainder term by integrating by parts. It would be slightly disagreeable to break this sequence.

Experience has shown that Chapters III through VIII make up an appropriate curriculum for one term (differentiation and elementary functions) while Chapters IX through XIII make up an appropriate curriculum for a second term (integration and Taylor's formula). The first two chapters may be used for a quick review by classes which are not especially well prepared.

I find that all these factors more than offset the possible disadvantage that for other courses (physics, chemistry perhaps) integration is needed early. This may be true, but so are the other topics, and unfortunately the course has to be projected in a totally ordered way on the time axis.

In addition to this, studying the log and exponential before integration has the advantage that we meet in a special concrete case the situation where we find an antiderivative by means of area: $\log x$ is the area under $1/x$ between 1 and x. We also see in this concrete case how $dA(x)/dx = f(x)$, where $A(x)$ is the area. This is then done again in full generality when studying the integral. Furthermore, inequalities involving lower sums and upper sums, having already been used in this concrete case, become more easily understandable in the general case. Classes which start the term on integration without having gone through the part on differentiation might well start with the last section of the chapter on logarithms, i.e. the last section of Chapter VIII.

Taylor's formula is proved with the integral form of the remainder, which is then properly estimated. The proof with integration by parts is more natural than the other (differentiating some complicated expression pulled out of nowhere), and is the one which generalizes to the higher dimensional case. I have placed integration after differentiation, because otherwise one has no technique available to evaluate integrals.

I personally think that the computations which arise naturally from Taylor's formula (computations of values of elementary functions, computation of e, π, $\log 2$, computations of definite integrals to a few decimals, traditionally slighted in calculus courses) are important. This was clear already many years ago, and is even clearer today in the light of the pocket computer proliferation. Designs of such computers rely precisely on effective means of computation by means of the Taylor polynomials. Learning how to estimate effectively the remainder term in Taylor's formula gives a very good feeling for the elementary functions, not obtainable otherwise.

The computation of integrals like

$$\int_0^1 e^{-x^2}\,dx \quad \text{or} \quad \int_0^{0.1} e^{-x^2}\,dx$$

which can easily be carried out numerically, without the use of a simple form for the indefinite integral, should also be emphasized. Again it gives a good feeling for an aspect of the integral not obtainable otherwise. Many texts slight these applications in favor of expanded treatment of applications of integration to various engineering situations, like fluid pressure on a dam, mainly by historical accident. I have nothing against fluid pressure, but one should keep in mind that too much time spent on some topics prevents adequate time being spent on others. For instance, Ron Infante tells me that numerical computations of integrals like

$$\int_0^1 \frac{\sin x}{x}\,dx,$$

which we carry out in Chapter XIII, occur frequently in the study of communication networks, in connection with square waves. Each instructor has to exercise some judgment as to what should be emphasized at the expense of something else.

The chapters on functions of several variables are included for classes which can proceed at a faster rate, and therefore have time for additional material during the first year. **Under ordinary circumstances, these chapters will not be covered during a first-year course. For instance, they are not covered during the first-year course at Yale.**

Induction. I think the first course in calculus is a good time to learn induction. However, an attempt to teach induction without having met natural examples first meets with very great psychological difficulties. Hence throughout the part on differentiation, I have not mentioned induction formally. Whenever a situation arises where induction may be used, I carry out stepwise procedures illustrating the inductive procedure. After enough repetitions of these, the student is then ready to see a pattern which can be summarized by the formal "induction," which just becomes a name given to a notion which has already been understood.

Review material. The present edition also emphasizes more review material. Deficient high school training is responsible for many of the difficulties experienced at the college level. These difficulties are not so much due to the problem of understanding calculus as to the inability to handle elementary algebra. A large group of students cannot automatically give the expansion for expressions like

$$(a + b)^2, \qquad (a - b)^2, \qquad \text{or} \qquad (a + b)(a - b).$$

The answers should be memorized like the multiplication table. To memorize by rote such basic formulas is not incompatible with learning general principles. It is complementary.

To avoid any misunderstandings, I wish to state explicitly that the poor preparation of so many high school students cannot be attributed to the "new math" versus the "old math." When I started teaching calculus as a graduate student in 1950, I found the quasi-totality of college freshmen badly prepared. Today, I find only a substantial number of them (it is hard to measure how many). On the other hand, a sizable group at the top has had the opportunity to learn some calculus, even as much as one year, which would have been inconceivable in former times. As bad as the situation is, it is nevertheless an improvement.

I wish to thank my colleagues at Yale and others in the past who have suggested improvements in the book: Edward Bierstone (University of Toronto), Folke Eriksson (University of Gothenburg), R. W. Gatterdam (University of Wisconsin, Parkside), and George Metakides (University of Rochester). I thank Ron Infante for assisting with the proofreading.

I am also much indebted to Anthony Petrello for checking worked-out examples and answers in past editions.

S. Lang

Part One

Review of Basic Material

If you are already at ease with the elementary properties of numbers and if you know about coordinates and the graphs of the standard equations (linear equations, parabolas, ellipses), then you should start immediately with Chapter III on derivatives.

CHAPTER I

Numbers and Functions

In starting the study of any sort of mathematics, we cannot prove everything. Every time that we introduce a new concept, we must define it in terms of a concept whose meaning is already known to us, and it is impossible to keep going backwards defining forever. Thus we must choose our starting place, what we assume to be known, and what we are willing to explain and prove in terms of these assumptions.

At the beginning of this chapter, we shall describe most of the things which we assume known for this course. Actually, this involves very little. Roughly speaking, we assume that you know about numbers, addition, subtraction, multiplication, and division (by numbers other than 0). We shall recall the properties of inequalities (when a number is greater than another). On a few occasions we shall take for granted certain properties of numbers which might not have occurred to you before and which will always be made precise. Proofs of these properties will be supplied in the Appendix for those of you who are interested.

I, §1. INTEGERS, RATIONAL NUMBERS, AND REAL NUMBERS

The most common numbers are the numbers $1, 2, 3, \ldots$ which are called **positive integers**.

The numbers $-1, -2, -3, \ldots$ are called **negative integers**. When we want to speak of the positive integers together with the negative integers and 0, we call them simply **integers**. Thus the integers are $0, 1, -1, 2, -2, 3, -3, \ldots$.

The sum and product of two integers are again integers.

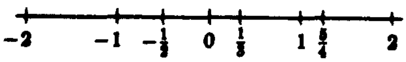
In addition to the integers we have **fractions**, like $\frac{3}{4}$, $\frac{5}{7}$, $-\frac{1}{8}$, $-\frac{101}{27}$, $\frac{8}{16}, \ldots$, which may be positive or negative, and which can be written as quotients m/n, where m, n are integers and n is not equal to 0. Such fractions are called **rational numbers**. Every integer m is a rational number, because it can be written as $m/1$, but of course it is not true that every rational number is an integer. We observe that the sum and product of two rational numbers are again rational numbers. If a/b and m/n are two rational numbers (a, b, m, n being integers and b, n unequal to 0), then their sum and product are given by the following formulas, which you know from elementary school:

$$\frac{a}{b}\frac{m}{n} = \frac{am}{bn},$$

$$\frac{a}{b} + \frac{m}{n} = \frac{an + bm}{bn}.$$

In this second formula, we have simply put the two fractions over the common denominator bn.

We can represent the integers and rational numbers geometrically on a straight line. We first select a unit length. The integers are multiples of this unit, and the rational numbers are fractional parts of this unit. We have drawn a few rational numbers on the line below.



Observe that the negative integers and rational numbers occur to the left of zero.

Finally, we have the numbers which can be represented by infinite decimals, like $\sqrt{2} = 1.414\ldots$ or $\pi = 3.14159\ldots$, and which will be called **real numbers** or simply **numbers**.

The integers and rational numbers are special cases of these infinite decimals. For instance,

$$3 = 3.000000\ldots,$$

and

$$\tfrac{3}{4} = 0.7500000\ldots,$$

$$\tfrac{1}{3} = 0.3333333\ldots.$$

We see that there may be several ways of denoting the same number, for instance as the fraction $\frac{1}{3}$ or as the infinite decimal $0.33333\ldots$. We have written the decimals with dots at the end. If we stop the decimal expansion at any given place, we obtain an approximation to the number. The further off we stop the decimal, the better approximation we obtain.

Finding the decimal expansion for a fraction is easy by the process of long division which you should know from high school.

Later in the course we shall learn how to find decimal expansions for other numbers which you may have heard about, like π. You were probably told that $\pi = 3.14\ldots$ but were not told why. You will learn how to compute arbitrarily many decimals for π in Chapter XIII.

Geometrically, the numbers are represented as the collection of all points on the above straight line, not only those which are a rational part of the unit length or a multiple of it.

We note that the sum and product of two numbers are numbers. If a is a number unequal to zero, then there is a unique number b such that $ab = ba = 1$, and we write

$$b = \frac{1}{a} \qquad \text{or} \qquad b = a^{-1}.$$

We say that b is the **inverse** of a, or "a inverse." We emphasize that the expression

$$\mathbf{1/0} \qquad \textbf{or} \qquad \mathbf{0^{-1}} \quad \textbf{is not defined.}$$

In other words, we cannot divide by zero, and we do not attribute any meaning to the symbols $1/0$ or 0^{-1}.

However, if a is a number, then the prodct $0 \cdot a$ is defined and is equal to 0. The product of any number and 0 is 0. Furthermore, if b is any number unequal to 0, then $0/b$ is defined and equal to 0. It can also be written $0 \cdot (1/b)$.

If a is a rational number $\neq 0$, then $1/a$ is also a rational number. Indeed, if we can write $a = m/n$, with integers m, n both different from 0, then

$$\frac{1}{a} = \frac{n}{m}$$

is also a rational number.

I, §2. INEQUALITIES

Aside from addition, multiplication, subtraction, and division (by numbers other than 0), we shall now discuss another important feature of the real numbers.

We have the **positive numbers**, represented geometrically on the straight line by those numbers unequal to 0 and lying to the right of 0. If a is a positive number, we write $a > 0$. You have no doubt already worked with positive numbers, and with inequalities. The next two properties are the most basic ones, concerning positivity.

POS 1. *If a, b are positive, so is the product ab and the sum $a + b$.*

POS 2. *If a is a number, then either a is positive, or $a = 0$, or $-a$ is positive, and these possibilities are mutually exclusive.*

If a number is not positive and not 0, then we say that this number is **negative**. By **POS 2**, if a is negative, then $-a$ is positive.

Although you know already that the number 1 is positive, it can in fact be **proved** from our two properties. It may interest you to see the proof, which runs as follows and is very simple. By **POS 2**, we know that either 1 or -1 is positive. If 1 is not positive, then -1 is positive. By **POS 1**, it must then follow that $(-1)(-1)$ is positive. But this product is equal to 1. Consequently, it must be 1 which is positive, and not -1. Using property **POS 1**, we could now conclude that $1 + 1 = 2$ is positive, that $2 + 1 = 3$ is positive, and so forth.

If $a > 0$, we shall say that a is **greater than** 0. If we wish to say that a is positive or equal to 0, we write

$$a \geqq 0$$

and read this "a greater than or equal to zero."

Given two numbers a, b we shall say that a is **greater than** b and write $a > b$ if $a - b > 0$. We write $a < 0$ (a is **less than** 0) if $-a > 0$ and $a < b$ if $b > a$. Thus $3 > 2$ because $3 - 2 > 0$.

We shall write $a \geqq b$ when we want to say that a is **greater than or equal to** b. Thus $3 \geqq 2$ and $3 \geqq 3$ are both true inequalities.

Other rules concerning inequalities are valid.

In what follows, let a, b, c be numbers.

Rule 1. *If $a > b$ and $b > c$, then $a > c$.*

Rule 2. *If $a > b$ and $c > 0$, then $ac > bc$.*

Rule 3. *If $a > b$ and $c < 0$, then $ac < bc$.*

Rule 2 expresses the fact that an inequality which is multiplied by a positive number is **preserved**. Rule 3 tells us that if we multiply both sides of an inequality by a negative number, then the inequality gets **reversed**. For instance, we have the inequality

$$1 < 3$$

Since $2 > 0$ we also have $2 \cdot 1 < 2 \cdot 3$. But -2 is negative, and if we multiply both sides by -2 we get

$$-2 > -6.$$

In the geometric representation of the real numbers on the line, -2 lies to the right of -6. This gives us the geometric representation of the fact that -2 is greater than -6.

If you wish, you may assume these three rules just as you assume **POS 1** and **POS 2**. All of these are used in practice. It turns out that the three rules can be proved in terms of **POS 1** and **POS 2**. We cannot assume all the inequalities which you will ever meet in practice. Hence just to show you some techniques which might recur for other applications, we show how we can deduce the three rules from **POS 1** and **POS 2**. You may omit these (short) proofs if you wish.

To prove Rule 1, suppose that $a > b$ and $b > c$. By definition, this means that $(a - b) > 0$ and $(b - c) > 0$. Using property **POS 1**, we conclude that

$$a - b + b - c > 0,$$

and canceling b gives us $(a - c) > 0$. By definition, this means $a > c$, as was to be shown.

To prove Rule 2, suppose that $a > b$ and $c > 0$. By definition,

$$a - b > 0.$$

Hence using the property of **POS 1** concerning the product of positive numbers, we conclude that

$$(a - b)c > 0.$$

The left-hand side of this inequality is none other than $ac - bc$, which is therefore > 0. Again by definition, this gives us

$$ac > bc.$$

We leave the proof of Rule 3 as an exercise.

We give an example showing how to use the three rules.

Example. Let a, b, c, d be numbers with $c, d > 0$. Suppose that

$$\frac{a}{c} < \frac{b}{d}.$$

We wish to prove the "cross-multiplication" rule that

$$ad < bc.$$

Using Rule 2, multiplying each side of the original inequality by c, we obtain

$$a < bc/d.$$

Using Rule 2 again, and multiplying each side by d, we obtain

$$ad < bc,$$

as desired.

Let a be a number > 0. Then there exists a number whose square is a. If $b^2 = a$ then we observe that

$$(-b)^2 = b^2$$

is also to a. Thus either b or $-b$ is positive. We agree to denote by $\sqrt{a}$ the **positive** square root and call it simply **the square root of** a. Thus $\sqrt{4}$ is equal to 2 and not -2, even though $(-2)^2 = 4$. This is the most practical convention about the use of the $\sqrt{}$ sign that we can make. Of course, the square root of 0 is 0 itself. A negative number does *not* have a square root in the real numbers.

There are thus two solutions to an equation

$$x^2 = a$$

with $a > 0$. These two solutions are $x = \sqrt{a}$ and $x = -\sqrt{a}$. For instance, the equation $x^2 = 3$ has the two solutions

$$x = \sqrt{3} = 1.732\ldots \qquad \text{and} \qquad x = -\sqrt{3} = -1.732\ldots.$$

The equation $x^2 = 0$ has exactly one solution, namely $x = 0$. The equation $x^2 = a$ with $a < 0$ has no solution in the real numbers.

Definition. Let a be a number. We define the **absolute value** of a to be

$$\boxed{|a| = \sqrt{a^2}.}$$

In particular,

$$\boxed{|a|^2 = a^2.}$$