CLASSICAL ELECTRICITY AND MAGNETISM

by

WOLFGANG K. H. PANOFSKY

MELBA PHILLIPS

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WOLFGANG K. H. PANOFSKY
Stanford University

and

MELBA PHILLIPS
Washington University

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PREFACE TO THE FIRST EDITION

This book is designed to emphasize those aspects of classical electricity and magnetism most useful to the modern student as a background both for experimental physics and for the quantum theory of matter and radiation. We have made no attempts at novelty beyond those inherent in looking at subject matter that has become a part of the foundations of physics, and has thus gained in usefulness as it has lost in immediacy. While no rigid adherence to historical development is attempted, the emphasis is on physical theory as evolved from fundamental empirical laws rather than on mathematics and strict internal logic. Thus Maxwell's equations are derived from the experimental laws of Coulomb, Ampère, and Faraday, instead of being postulated initially. In the opinion of the authors the physical concepts emerge more clearly in this way, and the approach represents the manner in which physical theory evolves in practice. The field formulation is preferred to the action-at-a-distance viewpoint even in electrostatics, however, since for the conventional treatment it is more readily extended to the nonstatic case. This despite the fact that it is possible, both for static and for nonstatic phenomena, to formulate an entirely consistent electromagnetic theory based on the delayed-action-at-a-distance principle.

The climax of 19th century electrodynamics was the theory of electromagnetic waves and its confirmation, and it is inevitable that any treatment of the subject today includes the principles of recent applications involving metallic boundaries. The introduction of the electrodynamic potentials and the Hertz solution of the wave equation are treated in the conventional way, but we have chosen to introduce the special theory of relativity before undertaking the theory of the electron. Historically the evidence was building up simultaneously along two separate lines, and many of the early difficulties in the derivation of radiation theory as applied to elementary charges were clarified in a very simple way by relativistic considerations. This approach has the advantage that the other problems of classical electron theory, especially those which have taken on added significance with the advent of quantum theory, can be exhibited more clearly.

Rationalized mks units are used throughout, simply because the majority of modern reference books and papers are now written in this system. Especially in the consideration of the electron, all quantities are so written that they can be immediately translated into Gaussian units. In

Appendix I will be found a discussion of the units in current use, and tables contain the fundamental relations of electrodynamics expressed in various systems as well as numerical conversion factors.

The text is based on graduate course lectures given by one of us (Panofsky) at the University of California and Stanford University. Early mimeographed notes on much of the subject matter were prepared with the aid of Howard Chang, Roger Wallace, Richard Madey, and Lee Aamodt, whose help is gratefully acknowledged. The editorial help of Miss Laurose Becker is also acknowledged with thanks.

The reader is assumed to have had courses in advanced calculus, differential equations, vector analysis, and, at least for the latter portions, is assumed to be familiar with classical mechanics on the graduate level. Prior knowledge of tensor analysis would be helpful, but is not necessary. References to appropriate collateral and background material are included at the end of each chapter, with some indication of what relevant material is to be found in each reference, and a full bibliographical list is given at the end.

The presentation is designed to be somewhat flexible, depending on the organization of course material. For purely theoretical courses Chapters 4 and 5, together with portions of other chapters dealing with partioular applications of potential theory, etc., may be omitted entirely. Some of the material in Chapter 12 is often covered in optics courses. And if a course in relativity theory is given separately Chapters 15–18 may be omitted, since we have endeavored to make Chapter 19 continuous with Chapter 14, insofar as the theory of radiation is concerned.

A final word about problems: for the most part they are designed to supplement the text. It had been our intention to give credit to original sources for those we did not invent ourselves, but in almost every case this turns out to be impossible: like discoveries, problems are rarely made singly, and in a subject as old as this ingenuity mainly recreates old ideas. And despite our adherence to the exhortation used by Becker, "be ye doers of the word and not hearers only, deceiving your own selves," we have not concentrated primarily on problem solving. The heart of the matter, we believe, lies in the ideas and their development.

W. K. H. P.
M. P.

PREFACE TO THE SECOND EDITION

The second edition of *Classical Electricity and Magnetism* is intended principally to remedy errors and inadequacies of the first edition. We have attempted to correct errors and make extensive revisions without changing the basic approach to the material; we hope that in so doing we have responded to the many helpful comments we have received from users of the book without introducing too many departures. The only radical change is in the treatment of radiation reaction, which has been completely rewritten and introduces new concepts. New material has been added in several instances: there is a new chapter on the basic principles of magnetohydrodynamics; the use of "superpotentials" for obtaining symmetric expansion of electric and magnetic wave-fields has been introduced; the material on the classical radiation of electrons moving in a circle has been expanded; the motion of particles with spin is treated; and the classical forms of such theorems as the dispersion relation and the "optical" or "shadow" theorem are now included.

We have not attempted to make the methods used in this book uniform; on the contrary, we believe that there is a great deal of educational value in the demonstration that many of the methods used are equivalent. As before, we stress physical ideas rather than mathematical techniques.

Without the generous help of many correspondents, who have pointed out errors or transmitted comments, this revision would not have been possible. This help has been so extensive that we cannot acknowledge each contribution; we are, however, particularly grateful to F. Rohrlich for a helpful exchange of correspondence. We are also much indebted to Mrs. Laurose Richter for assistance in preparing the manuscript and to Mrs. Adèle Panofsky for preparing the index.

W. K. H. P.
M. P.

Stanford and St. Louis

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CHAPTER 1

THE ELECTROSTATIC FIELD IN VACUUM

The interaction between material bodies can be described either by formulating the action at a distance between the interacting bodies or by separating the interaction process into the production of a *field* by one system and the action of the field on another system. These two alternative descriptions are physically indistinguishable in the static case. If the bodies are in motion, however, and the velocity of propagation of the interaction is finite, it is both physically and mathematically advantageous to ascribe physical reality to the field itself, even though it is possible to replace the field concept by that of "delayed" and "advanced" direct interaction in the description of electromagnetic phenomena. We shall formulate even the electrostatic interactions as a field theory, which can then be extended to the consideration of nonstatic cases.

1–1 Vector fields. Field theories applicable to various types of interaction differ by the number of parameters necessary to define the field and by the symmetry character of the field. In a general sense, a field is a physical entity such that each point in space is a degree of freedom. A field is therefore specified by giving the behavior in time at each coordinate point of a quantity suitable to describe the physical content.

The types of fields possible are restricted by various considerations. Fields are classified according to the number of parameters necessary to define the field and by the "transformation character" of the field quantities under various coordinate transformations. A "scalar" field is described by the time dependence of one quantity at each point in space, a "three-dimensional vector field" by three such quantities. In general, an "nth-rank tensor field" requires the specification of d^n components, where d is the dimensionality of the space in which the field is defined. A scalar field is a zero-rank tensor field, and a vector field is a first-rank tensor field.

The field description of a physical entity is independent of the particular choice of coordinate system used. This fact restricts the transformation properties of the field components under coordinate transformations. We consider two types of transformations of coordinates, "proper" and "improper" transformations. Proper transformations are those which leave the cyclic order of the coordinates invariant (i.e., do not transform a right-handed into a left-handed coordinate system in three dimensions);

translation and rotation are proper transformations. Improper transformations, such as inversion of the coordinate axes and reflection of the coordinate system in a plane, change the cyclic order of coordinates.

A basic vector is the distance $\mathbf{r}$ connecting two points; the components of $\mathbf{r}$ may be designated by r_α. The components V_α of a *vector* field $\mathbf{V}$ transform like the components r_α under both proper and improper transformations. A *scalar* is invariant under proper and improper transformations. The components P_α of a *pseudovector* field $\mathbf{P}$ transform like the components r_α under proper transformations, but change sign relative to r_α under improper transformations. A *pseudoscalar* is invariant under proper transformations but changes sign under improper transformations.

The electric field is a three-dimensional vector field, i.e., a field definable by the specification of three components. The theory of vector fields was developed in connection with the study of fluid motion, a fact which is betrayed repeatedly by the vocabulary of the theory. We shall consider some general mathematical properties of such fields before specifying the physical content of the vectors.

All vector fields in three dimensions are uniquely defined if their circulation densities (curl) and source densities (divergence) are given functions of the coordinates at all points in space, and if the totality of sources, as well as the source density, is zero at infinity. Let us prove this theorem formally. Consider a three-dimensional vector field $\mathbf{V}(x, y, z)$ such that

$$\nabla \cdot \mathbf{V} = s, \tag{1-1}$$

$$\nabla \times \mathbf{V} = \mathbf{c}. \tag{1-2}$$

Equation (1–2) is self-consistent only if the circulation density $\mathbf{c}$ is irrotational, i.e., if

$$\nabla \cdot \mathbf{c} = 0. \tag{1-2$'$}$$

We shall first show that if

$$\mathbf{V} = -\nabla\phi + \nabla \times \mathbf{A}, \tag{1-3}$$

where

$$\phi(x_\alpha) = \frac{1}{4\pi}\int \frac{s(x'_\alpha)}{r(x_\alpha, x'_\alpha)}\, dv' \tag{1-4}$$

and

$$\mathbf{A}(x_\alpha) = \frac{1}{4\pi}\int \frac{\mathbf{c}(x'_\alpha)}{r(x_\alpha, x'_\alpha)}\, dv', \tag{1-5}$$

then $\mathbf{V}$ satisfies Eqs. (1–1) and (1–2).

It is necessary to examine the notation of Eqs. (1-4) and (1-5) before proceeding with the proof. The symbol x_α stands for x, y, z at the *field point;* the symbol x'_α stands for x', y', z' at the *source point;* the function $r(x_\alpha, x'_\alpha)$ is the symmetric function

$$r(x_\alpha, x'_\alpha) = \left| \sqrt{\sum_{\alpha=1}^{\alpha=3} (x_\alpha - x'_\alpha)^2} \right|$$

representing the positive distance between field and source point. The reader should note carefully the functional relationships explicit in Eqs. (1-4) and (1-5). In integrals of this type these functional dependences will often not be fully stated; for example, we may write the volume integrals

$$\phi = \frac{1}{4\pi} \int \frac{s}{r}\, dv', \tag{1-4'}$$

$$\mathbf{A} = \frac{1}{4\pi} \int \frac{\mathbf{c}}{r}\, dv', \tag{1-5'}$$

as a short notation. We shall sometimes use $\mathbf{R}$ for the radius vector from an origin of coordinates to the field point x_α, and $\boldsymbol{\xi}$ for that of a source point x'_α; then $r = |\mathbf{R} - \boldsymbol{\xi}|$.

Let us demonstrate that $\mathbf{V}$ as expressed by Eq. (1-3) is a solution of Eqs. (1-1) and (1-2):

$$\begin{aligned} \nabla \cdot \mathbf{V} &= -\nabla^2\phi + \nabla \cdot (\nabla \times \mathbf{A}) = -\nabla^2\phi \\ &= -\frac{1}{4\pi} \nabla^2 \left\{ \int \frac{s}{r}\, dv' \right\}. \end{aligned}$$

The Laplacian operator ∇^2 operates on the field coordinates; hence

$$\nabla \cdot \mathbf{V} = -\frac{1}{4\pi} \int s \nabla^2 \left(\frac{1}{r}\right) dv'. \tag{1-6}$$

Now we can show that

$$\nabla^2 \left\{ \frac{1}{r(x_\alpha, x'_\alpha)} \right\} = -4\pi\, \delta(\mathbf{r}), \tag{1-7}$$

where $\delta(\mathbf{r})$, the Dirac δ-function, is defined by the functional properties

$$\delta(\mathbf{r}) = 0, \qquad \mathbf{r} \neq 0, \quad \text{i.e.,} \quad x_\alpha \neq x'_\alpha, \tag{1-8}$$

$$\int \delta(\mathbf{r})\, dv' = 1, \tag{1-9}$$

if the point $\mathbf{r} = 0$ is included in the volume of integration, and by

$$\int f(x'_\alpha)\, \delta(\mathbf{r})\, dv' = f(x_\alpha), \tag{1-10}$$

for any arbitrary function f so long as the volume of integration includes the point $\mathbf{r} = 0$. The δ-function is not an analytic function but essentially a notation for the functional properties of the three defining equations. It will always be used in terms of these properties.

Since it is evident by direct differentiation that $\nabla^2(1/r) = 0$ for $r \neq 0$, we have only to prove that

$$\int \nabla^2(1/r)\, dv' = -4\pi \tag{1-11}$$

in order to verify Eq. (1–7). [In Eq. (1–11) the point $r = 0$, that is, $x_\alpha = x'_\alpha$, is included in the volume of integration.] By the application of Gauss's divergence theorem, applicable to any vector $\mathbf{V}$,*

$$\int \nabla \cdot \mathbf{V}\, dv = \int \mathbf{V} \cdot d\mathbf{S},$$

it is seen that

$$\int \nabla^2 \left(\frac{1}{r}\right) dv' = \int \nabla \left(\frac{1}{r}\right) \cdot d\mathbf{S}'$$

$$= -\int \frac{\mathbf{r} \cdot d\mathbf{S}'}{r^3} = -\int d\Omega,$$

where Ω is the solid angle subtended at x_α by the surface of integration S' over the variables x'_α. Since S' includes x_α, we have simply $\int d\Omega = 4\pi$, and Eq. (1–11) is verified. Hence from Eqs. (1–6) and (1–10),

$$\nabla \cdot \mathbf{V} = -\frac{1}{4\pi} \int s \nabla^2 \left(\frac{1}{r}\right) dv' = \int s(x'_\alpha)\, \delta(\mathbf{r})\, dv' = s(x_\alpha), \tag{1-12}$$

which was to be proved.

---

* Strictly speaking, Gauss's divergence theorem is not necessarily applicable, since the function $\mathbf{V} = \nabla(1/r)$ is singular at $r = 0$. If, however, we remove the singularity by substituting for $1/r$ the function $(1 - e^{-r/a})/r$, for example, where a is an arbitrarily small radius, then

$$\nabla \left[\frac{1}{r}(1 - e^{-r/a})\right] = -\frac{\mathbf{r}}{r^3}(1 - e^{-r/a}) + \frac{\mathbf{r}}{r^3}\left(\frac{r}{a} e^{-r/a}\right).$$

Since the magnitude of the second term varies only as r^{-1}, its surface integral over a small sphere surrounding the point $r = 0$ will vanish as the radius of the sphere goes to zero.

Similarly,

$$\nabla \times \mathbf{V} = -\nabla \times \nabla\phi + \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$$

$$= \frac{1}{4\pi}\left\{\int (\mathbf{c} \cdot \nabla)\nabla\left(\frac{1}{r}\right) dv' - \int \mathbf{c}\nabla^2\left(\frac{1}{r}\right) dv'\right\}. \tag{1–13}$$

We shall be able to show that the first integral vanishes if **c** is bounded in space. If we anticipate this result, we see immediately, from Eq. (1–7), that

$$\nabla \times \mathbf{V} = \int \mathbf{c}(x'_\alpha)\, \delta(\mathbf{r})\, dv' = \mathbf{c}(x_\alpha), \tag{1–14}$$

so that Eq. (1–2) is also satisfied.

To prove that the first term of Eq. (1–13) vanishes, let us examine the coordinate variables involved in the integrand. The operator ∇ has the components $\partial/\partial x_\alpha$. If we introduce the operator $\nabla'_\alpha = \partial/\partial x'_\alpha$, operating on the source coordinates, then for any arbitrary function $g[r(x_\alpha, x'_\alpha)]$, we have

$$\nabla g = -\nabla' g. \tag{1–15}$$

Therefore the first integral of Eq. (1–13) may be written

$$\mathbf{I} = \int (\mathbf{c} \cdot \nabla)\nabla\left(\frac{1}{r}\right) dv' = \int (\mathbf{c} \cdot \nabla')\nabla'\left(\frac{1}{r}\right) dv'.$$

The differential operators now operate on the variables of integration and we may integrate by parts. Each component of **I** becomes

$$I_\alpha = \int (\mathbf{c} \cdot \nabla')\frac{\partial}{\partial x'_\alpha}\left(\frac{1}{r}\right) dv'$$

$$= \int \nabla' \cdot \left\{\mathbf{c}\frac{\partial}{\partial x'_\alpha}\left(\frac{1}{r}\right)\right\} dv' - \int (\nabla' \cdot \mathbf{c})\frac{\partial}{\partial x'_\alpha}\left(\frac{1}{r}\right) dv'. \tag{1–16}$$

The second integral vanishes because the divergence of **c** is zero [Eq. (1–2′)]. The first term can be transformed to a surface integral by means of Gauss's theorem; if **c** is bounded in space the surface may be taken sufficiently large so that **c** is zero over the entire integration. Hence Eq. (1–16) is zero, and the proof is complete.

We have thus proved that if the source density s and the circulation density **c** of a vector field **V** are given everywhere, then a solution for **V** can be derived from a *scalar potential* ϕ and a *vector potential* **A**. The potentials ϕ and **A** are expressed as integrals over the source and circulation densities.