Modelling and Performance Evaluation Methodology

Edited by F. Baccelli and G. Fayolle

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Preface

The purpose of this international seminar was to bring together promising probabilistic tools and advanced research in computer system performance evaluation and, more generaly, to contribute to the enlargment of the interface between computer science and probability.

The computer science oriented papers collected in this volume cover a wide range of applications including communication systems, architecture, data structures and algorithms.

Salient features of these systems such as throuput, response time or stability condition can be formulated in term of probabilistic problems of specific nature.

These problems, in turn, often reduce to the analysis of some steady state properties of basic continuous (or point) processes related to queueing models.

The main mathematical methods -which rely on Markovian features, functional equations and ergodic theory- are presented in more theoretical or survey papers.

F. Baccelli G. Fayolle

Preface

L'objectif de ce séminaire international était de mettre en contact des outils probabilistes prometteurs et les recherches sur l'évaluation de performances des systèmes informatiques.

Les articles "informatiques" continus dans ce volume concernent de nombreux domaines d'applications. On trouvera notamment des analyses de systèmes de communication, d'architectures, de structures de données ou d'algorithmes.

On peut voir dans le détail dans quelle mesure la détermination de caractéristiques essentielles de ces systèmes -tels que le débit, le temps de réponse ou la condition de stabilité- peut se formuler comme un problème probabiliste.

A leur tour, ces problèmes se réduisent le plus souvent à l'analyse de certaines propriétés stationnaires de processus continus ou ponctuels spécifiques liés à des modèles de files d'attente.

Les outils probabilistes les plus utilisés, qui reposent sur l'analyse markovienne, les méthodes d'équations fonctionnelles et la théorie ergodique, sont aussi présentés dans des articles de nature plus théorique.

F. BACCELLI G. FAYOLLE

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AN ASYMPTOTIC ANALYSIS OF BLOCKING

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1. INTRODUCTION

Product form queueing networks have proved valuable in modelling a variety of computer and communication systems, and have been flexible enough to represent adequately many of the features arising in such applications ([2], [3], [6]). They have not, however, been able to provide much insight into the phenomenon of blocking, a phenomenon which appears particularly unyielding to any general form of exact analysis. In this paper we outline the progress made with alternative, asymptotic approaches to blocking.

The approach to be discussed is based on a model which can be described as follows. Messages are transmitted through a series of nodes linked by communication channels. The lengths of successive messages are independent identically distributed random variables, and the time taken to transmit a message through a channel is equal to its length. Each node has a finite buffer, and when the number of messages at a node reaches the buffer size transmission from the preceding node is interrupted. The most basic measure of the performance of such a system is the maximum rate at which it can accept messages, which we term the throughput. A system's throughput is in general difficult to calculate exactly, but there are available fairly tractable bounds. For a given system these bounds are not especially tight, but they do make possible a number of qualitative insights into the phenomenon of blocking. In particular, we discuss the rate of decay of throughput as series length increases, and the rate at which buffer sizes should grow to ensure that throughput does not decline to zero. We also consider the effect of reordering the sequence of messages, and the effect of segregating long messages from the rest.

The reader is referred to [4] and [5] for further discussion of the topics of this paper and for detailed proofs of the results quoted here, and to [1] and [7] for reviews of previous work on queueing systems with blocking.

2. THROUGHPUT AND SERIES LENGTH

An infinite sequence of messages is to be transmitted through a series of n nodes linked by communication channels (Figure 1). A message is not available for transmission from node i $(i=2,3,\ldots,n)$ until its transmission from node $i-1$ has been completed, and each node transmits messages in the order of their arrival. Each node has a buffer able to hold up to B messages. If node i contains B messages then transmission from node $i-1$ is blocked and must wait until node i has completed transmission of a message. Input of messages to node 1 is instantaneous, so that this node always contains B messages, and transmission from node n is never blocked. The time taken by a node to transmit a message is equal to the length of the message, and is thus the same at each of the n nodes. The length of the u^{th} message is X_u, where $X_1, X_2, \ldots$ are a sequence of independent positive random variables with common distribution F.

Suppose that at time $t=0$ the system begins operation with nodes $2,3,\ldots,n$ empty and the first B messages of the input sequence present at node 1. Let N_t be the number of messages which have completed transmission from node n by time t. Then we define the throughput of the system to be

$$\lambda(n,B,F) = \lim_{t\to\infty} \frac{EN_t}{t}$$

$$= \lim_{t\to\infty} \frac{N_t}{t} \qquad \text{a.s.}$$

where the existence and equality of the limits can be demonstrated. It will be convenient to use the symbol $\lambda(n,B,F)$ as a label for the system itself as well as for the numerical value of the system's throughput.

Although it is in general difficult to calculate $\lambda(n,B,F)$, bounds can be obtained quite easily. For example, consider the system $\lambda(n,1,F)$. The throughput of this system can only be improved if node i is allowed to begin, but not complete, transmission of a message when node $i+1$ is full. This corresponds to the rule common in models of manufacturing job-shops where the server at node i can process a job even though node $i+1$ is full, but the job cannot move on and release the server at node i until there is a space available at node $i+1$. Now the time taken to input a message to the amended system is simply the maximum of the previous n message lengths. Hence the throughput of the amended system is $M(n,F)^{-1}$, where

$$M(n,F) = E\max\{X_1, X_2, \ldots, X_n\}$$

$$= \int_0^\infty [1 - F(x)^n]dx \ .$$

We thus have the upper bound

$$\lambda(n,1,F) \le M(n,F)^{-1} \ . \tag{2.1}$$

A lower bound can be found by a related argument. Suppose that operation of the system $\lambda(2n,1,F)$ is restricted as follows: no message can complete transmission from node i $(i=1,2,\ldots,2n-2)$ until node $i+2$ (in addition to node $i+1$) is empty. For the restricted system the time that elapses between message $u-1$ and message u $(u>n)$ leaving node 1 is $\max\{X_{u-1},X_{u-2},\ldots,X_{u-n}\}$ $+ \max\{X_u,X_{u-1},\ldots,X_{u-n+1}\}$ and so the throughput of the restricted system is $[2M(n,F)]^{-1}$. Thus

$$\lambda(2n,1,F) \ge \tfrac{1}{2}M(n,F)^{-1} \tag{2.2}$$

To illustrate these bounds suppose that message lengths are exponentially distributed, with

$$F(x) = 1 - e^{-x} \qquad x \ge 0 \ .$$

Then

$$M(n,F) = \sum_{i=1}^{n} \frac{1}{i}$$

and so

$$\left(2\sum_{i=1}^{n} \frac{1}{i}\right)^{-1} \le \lambda(2n,1,F) \le \left(\sum_{i=1}^{2n} \frac{1}{i}\right)^{-1} \ . \tag{2.3}$$

Thus

$$\lambda(n,1,F) = 0((\log n)^{-1}) \tag{2.4}$$

where the notation $g(n) = 0(h(n))$ indicates that

$$0 < \liminf_{n\to\infty} \frac{g(n)}{h(n)} \le \limsup_{n\to\infty} \frac{g(n)}{h(n)} < \infty$$

Although simpler to state, the order relationship (2.4) gives a good deal less information than the bounds (2.3). It is worth noting that when in this paper we give order relationships such as (2.4) they are obtained from explicit bounds such as (2.3) whose calculation depends on the precise form of the distribution F .

The order relationship (2.4) can be generalized to values of B other than one, and to arbitrary distributions: the result is as follows.

<u>Theorem 2.1</u>

$$\lambda(n,B,F) = 0(\zeta(n,F)^{-1})$$

where

$$\zeta(n,F) = \inf\{K : K^{-1}\int_K^\infty x\,dF(x) \le \frac{1}{n}\}$$

The theorem puts no conditions on the distribution F , but care is needed in interpreting certain extreme cases. It is clear that if F has infinite mean then $\lambda(n,B,F) = 0$ and $\zeta(n,F) = \infty$, while if there is an upper bound M on message lengths, so that $F(M) = 1$ and $F(x) < 1$ for $x<M$, then as n tends to infinity $\lambda(n,B,F)$ decreases to a limit value not less than $\frac{1}{2}M^{-1}$ (or M^{-1} if $B\ge 2$).

The infimum is necessary in the definition of $\zeta(n,F)$ since the distribution F may have atoms: when the function F is continuous $\zeta = \zeta(n,F)$ satisfies

$$\int_\zeta^\infty x dF(x) = \frac{\zeta}{n} .$$

Suppose, for example,

$$1-F(x) = \frac{2}{\sqrt{2\pi}} \int_x^\infty e^{-z^2/2} dz , \tag{2.5}$$

so that X_1 is distributed as $|Y|$ where Y has a standard normal distribution. Then

$$\int_K^\infty x dF(x) = \frac{2}{\sqrt{2\pi}} e^{-K^2/2}$$

and so

$$\zeta^2 = 2 \log \left(\frac{n}{\zeta} \sqrt{\frac{2}{\pi}}\right)$$

Thus

$$\zeta(n,F) = 0(\sqrt{\log n})$$

and

$$\lambda(n,B,F) = 0((\sqrt{\log n})^{-1})$$

The following Corollaries deal with two important classes of distribution. Corollary 2.3 includes the exponential and normal examples discussed above.

<u>Corollary 2.2</u> If

$$1 - F(x) = 0(x^{-\rho})$$

where $\rho>1$ then

$$\lambda(n,B,F) = O(n^{-1/\rho})$$

Corollary 2.3 If

$$-\log[1 - F(x)] = O(x^{\rho})$$

then

$$\lambda(n,B,F) = O((\log n)^{-1/\rho})$$

3. THROUGHPUT AND BUFFER SIZE

To assess the effect of increasing the buffer size B it is necessary to obtain a further set of bounds on the throughput $\lambda(n,B,F)$. In this Section we indicate the arguments needed.

We begin by considering a system $\lambda(n,B,G)$ in which the distribution G concentrates probability one on the interval $[q,(B-1)q]$. Thus the longest possible message is no more than $(B-1)$ times the length of the shortest possible message. For this system the buffer at node n can never be full - even when $X_u = (B-1)q$ and $X_{u+1} = X_{u+2} = \ldots = X_{u+B-1} = q$ node n must be empty when message u arrives, and a time $(B-1)q$ later it just completes transmission of message u as its buffer receives message $u+B-1$. Thus transmission from node $n-1$ is never blocked, and an inductive argument shows that transmission is never blocked from any node. The throughput of this system is therefore maximal:

$$\lambda(n,B,G) = \left[\int_q^{(B-1)q} x dF(x)\right]^{-1} .$$

For an arbitrary distribution F the throughput $\lambda(n,B,F)$ can be bounded by a comparison with a system of the above form. Message lengths generated by F which are less than q are simply increased to q ; and, to deal with messages of length greater than