G. N. BERMAN

A Problem Book in Mathematical Analysis

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Г. Н. БЕРМАН

СБОРНИК ЗАДАЧ ПО КУРСУ МАТЕМАТИЧЕСКОГО АНАЛИЗА

ИЗДАТЕЛЬСТВО «НАУКА»

ИЗДАТЕЛЬСТВО · МИР ·

Preface

This collection of problems is designed for students studying mathematical analysis in higher technical educational institutions. Problems and exercises are systematically selected and arranged in compliance with the major sections of the course in mathematical analysis.

Theoretical information is not included in this book. The reader will find it in the corresponding sections of the textbook *Mathematical Analysis* (*A Brief Course for Engineering Students*) by A. F. Bermant and I. G. Aramanovich brought out in English by Mir Publishers of Moscow in 1975. For convenience, most sections of the present book are subdivided into parts. Necessary information is given before problems on physics. More difficult problems (indicated by an asterisk) are supplied with hints for their solution to be found in the Answers.

Tables of the values of basic elementary functions compiled by A. T. Tsvetkov are presented in the Appendix. The tables are borrowed from *Problems in Higher Mathematics* by V. P. Minorsky (Mir Publishers, Moscow, 1975).

The first edition of the present book came out in 1947. All subsequent editions, including two major revisions, were published without Georgii Nikolaevich Berman, who died February 9, 1949 after a long and severe illness acquired when he was wounded at the front during the Great Patriotic War. The work was undertaken jointly by Berman's colleagues I. G. Aramanovich, A. F. Bermant, B.A. Kordemsky, R. I. Pozoisky, and M. G. Shestopal.

Our collective lost its co-author and the editor of the first edition of the present book, Professor Anisim Fedorovich Bermant, who died suddenly May 26, 1959.

Georgii Nikolaevich and Anisim Fedorovich were esteemed colleagues, cultured men, and talented progressive teachers. They leave a lasting impression.

I. G. Aramanovich
B. A. Kordemsky
R. I. Pozoisky
M. G. Shestopal

Contents

Chapter I

Function

§ 1. Preliminaries

Representation of functions

1. The sum of the interior ang les of a plane convex polygonal is a function of the number of its sides. Represent this function analytically. What values can the argument attain?
2. A function y of x is specified by the following table:

Independent variable x	0	0.5		1	1.5	2	3
Function y	−1.5	−1		0	3.2	2.6	0
Independent variable x	4	5	6	7	8	9	10
Function y	−1.8	−2.8	0	1.1	1.4	1.9	2.4

Plot the graph of the function $y(x)$ by joining the points with a smooth line, and, using the graph thus obtained, make the given table "denser" by determining the values of the function for $x = 2.5, 3.5, 4.5, 5.5, 6.5, 7.5, 8.5, 9.5$.

3. A function is represented by the graph shown in Fig. 1. Draw the graph to a given scale on the graph paper. Take from the graph the values of the function corresponding to the chosen values of the independent variable, and tabulate these values.
4. A function is specified by the graph shown in Fig. 2. Taking advantage of the graph, answer the following questions:
(a) At what values of the independent variable does the function vanish?
(b) For what values of the independent variable is the function positive?

(c) For what values of the independent variable is the function negative?

5. The force F of electrostatic interaction of two point electric charges e_1 and e_2 is related to the distance r between them by the formula (Coulomb's law)

$$F = \frac{e_1 \cdot e_2}{\varepsilon \cdot r^2}.$$

Putting $e_1 = e_2 = 1$ and $\varepsilon = 1$, make a table of values of the given function for $r = 1, 2, 3, \ldots, 10$, and plot it joining the found points with a smooth line.

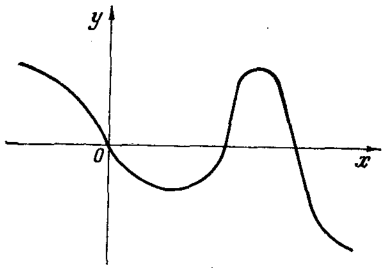

Fig. 1

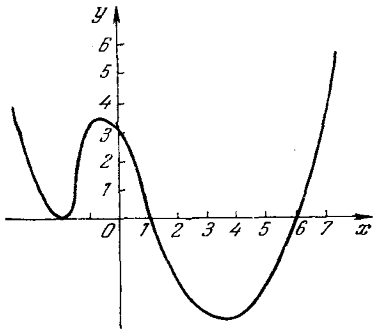

Fig. 2

6. Write the function expressing the relationship between the radius r of the cylinder base and its height h for a given volume $V = 1$. Compute the respective values of r for the following values of h: 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5. Plot the graph of the function.

7. Express the area of an isosceles trapezium with the bases a and b as a function of the angle α at the base a. Plot the graph of the function for $a = 2$, $b = 1$.

8. Express the length b of one side of a right-angled triangle as a function of the length a of the other at a constant hypotenuse $c = 5$. Graph this function.

9. Given the functions $f(x) = \frac{x-2}{x+1}$ and $\varphi(x) = \frac{|x-2|}{x+1}$.

Find: $f(0)$, $f(1)$, $f(2)$, $f(-2)$, $f\left(-\frac{1}{2}\right)$, $f(\sqrt{2})$, $\left|f\left(\frac{1}{2}\right)\right|$; $\varphi(0)$, $\varphi(1)$, $\varphi(2)$, $\varphi(-2)$, $\varphi(4)$. Are $f(-1)$ and $\varphi(-1)$ existant?

10. Given the function $f(u) = u^3 - 1$. Find: $f(1)$, $f(a)$, $f(a+1)$, $f(a-1)$, $2f(2a)$.

11. Given the functions $F(z) = 2^{z-2}$ and $\varphi(z) = 2|z|^{-2}$. Find: $F(0)$, $F(2)$, $F(3)$, $F(-1)$, $F(2.5)$, $F(-1.5)$ and $\varphi(0)$, $\varphi(2)$, $\varphi(-1)$, $\varphi(x)$, $\varphi(-1) + F(1)$.

12. Given the function $\psi(t) = t \cdot a^t$. Find $\psi(0)$, $\psi(1)$, $\psi(-1)$, $\psi\left(\frac{1}{a}\right)$, $\psi(a)$, $\psi(-a)$.

13. $\varphi(t) = t^3 + 1$. Find $\varphi(t^2)$ and $[\varphi(t)]^2$.

14. $F(x) = x^4 - 2x^2 + 5$. Prove that $F(a) = F(-a)$.

15. $\Phi(z) = z^3 - 5z$. Prove that $\Phi(-z) = -\Phi(z)$.

16. $f(t) = 2t^2 + \frac{2}{t^2} + \frac{5}{t} + 5t$. Prove that $f(t) = f\left(\frac{1}{t}\right)$.

17. $f(x) = \sin x - \cos x$. Prove that $f(1) > 0$.

18. $\psi(x) = \log_{10} x$. Prove that $\psi(x) + \psi(x+1) = \psi[x(x+1)]$.

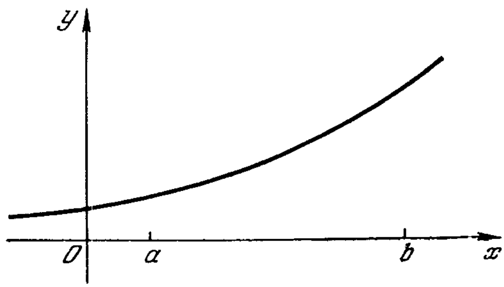

Fig. 3

19. $F(z) = a^z$. (1) Prove that the following relationship holds true for any z

$$F(-z) \cdot F(z) - 1 = 0.$$

(2) Prove that

$$F(x) \cdot F(y) = F(x+y).$$

20. Given: the graph of the function $y = f(x)$ and the values a and b of the independent variable x (Fig. 3). Construct the ordinates for $f(a)$ and $f(b)$. Give the geometrical meaning of the ratio $\frac{f(b) - f(a)}{b - a}$.

21. Show that if any chord on the graph of the function $y = f(x)$ lies higher than the subtended arc, then there exists the following inequality:

$$\frac{f(x_1) + f(x_2)}{2} > f\left(\frac{x_1 + x_2}{2}\right)$$

for all $x_1 \neq x_2$.

22. Given: $f(x) = x^2 - 2x + 3$. Find all roots of the equation (a) $f(x) = f(0)$; (b) $f(x) = f(-1)$.

23. Given: $f(x) = 2x^3 - 5x^2 - 23x$. Find all roots of the equation $f(x) = f(-2)$.

24. Given the function $f(x)$. Find at least one root of the equation $f(x) = f(a)$.

25. Find two roots of the equation $f(x) = f\left(\frac{x+8}{x-1}\right)$ if it is known that the function $f(x)$ is defined on the interval $[-5, 5]$. Find all roots of the given equation for the case when $f(x) = x^2 - 12x + 3$.

26. $F(x) = x^2 + 6$; $\varphi(x) = 5x$. Find all roots of the equation $F(x) = |\varphi(x)|$.

27. $f(x) = x + 1$; $\varphi(x) = x - 2$. Solve the equation

$$|f(x) + \varphi(x)| = |f(x)| + |\varphi(x)|.$$

28. In the expression of the function $f(x) = ax^2 + bx + 5$ find the values of a and b for which the identity $f(x+1) - f(x) \equiv 8x + 3$ is true.

29. Let $f(x) = a \cdot \cos(bx + c)$. For what values of the constants a, b, and c is the identity $f(x+1) - f(x) \equiv \sin x$ fulfilled?

Composite functions

30. Given: $y = z^2$, $z = x + 1$. Express y as a function of x.

31. Given: $y = \sqrt{z+1}$, $z = \tan^2 x$. Express y as a function of x.

32. Given: $y = z^2$, $z = \sqrt[3]{x+1}$, $x = a^t$. Express y as a function of t.

33. Given: $y = \sin x$, $v = \log_{10} y$, $u = \sqrt{1 + v^2}$. Express v as a function of x.

34. Given: $y = 1 + x$, $z = \cos y$, $v = \sqrt{1 - z^2}$. Express v as a function of x.

35. Represent the following composite functions by means of chains made of basic elementary functions:
(1) $y = \sin^3 x$; (2) $y = \sqrt[3]{(1+x)^2}$; (3) $y = \log_{10} \tan x$; (4) $y = \sin^3 (2x + 1)$; (5) $y = 5^{(3x+1)^2}$.

36. $f(x) = x^3 - x$; $\varphi(x) = \sin 2x$. Find:
(a) $f\left[\varphi\left(\frac{\pi}{12}\right)\right]$; (b) $\varphi[f(1)]$; (c) $\varphi[f(2)]$; (d) $f[\varphi(x)]$; (e) $f[f(x)]$; (f) $\{f[f(1)]\}$; (g) $\varphi[\varphi(x)]$.

37. Prove the validity of the following method of constructing the graph of the composite function $y = f[\varphi(x)] = F(x)$

given the graphs of its components: $y = f(x)$, $y = \varphi(x)$. From point A of the graph of the function $\varphi(x)$ (see Fig. 4) corresponding to the given value of the independent variable x draw a straight line parallel to the x-axis to intersect the bisector of the first and third quadrants at point B. From the point B draw a straight line parallel to the y-axis to intersect the graph of the function $f(x)$ at point C. If now a straight line is drawn from the point C and parallel to the x-axis, then point D at which it intersects the straight line NN' will be the point on the curve for the function $F(x)$ corresponding to the taken value of x.

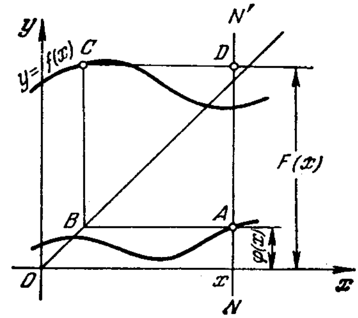

Fig. 4

Implicit functions

38. Write the explicit form of the function y represented implicitly by the following equations:

(1) $x^2 + y^2 = 1$; (2) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; (3) $x^3 + y^3 = a^3$; (4) $xy = C$; (5) $2^{xy} = 5$; (6) $\log_{10} x + \log_{10}(y + 1) = 4$; (7) $2^{x+y}(x^2 - 2) = x^3 + 7$; (8) $(1 + x)\cos y - x^2 = 0$.

39*. Show that for $x > 0$ the equation $y + |y| - x - |x| = 0$ defines the function whose graph is the bisector of the first quadrant, and for $x \leqslant 0$ the given equation is satisfied by the coordinates of all points of the third quadrant (including its boundary points).

§ 2. Simplest Properties of Functions

Domain of definition of a function

40. Tabulate the values of the function of an integral argument $y = \frac{1}{x!}$ for $1 \leqslant x \leqslant 6$.

41. The value of the function of an integral argument $u = f(n)$ is equal to the number of simple numbers not exceeding n. Compile a table of values of u for $1 \leqslant n \leqslant 20$.

42. The value of the function of an integral argument $u = f(n)$ is equal to the number of integral devisors, different from 1 and n, of the argument. Make a table of the values of u for $1 \leqslant n \leqslant 20$.

43. A bar (Fig. 5) is made up of three parts whose lengths are equal to 1, 2, 1 length units, their weights being equal to 2, 3, 1 weight units, respectively. The weight of the variable

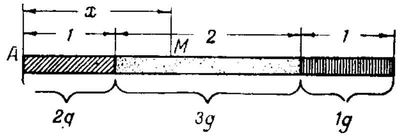

Fig. 5

part AM (Fig. 5) is a function of its length x. For what values of x is this function defined? Express it analytically and sketch its graph.

44. A tower has the following shape: a truncated right circular cone with the radii $2R$ (the lower base) and R (the upper base), and the height R, bears a right circular cylinder whose radius is R, the height being $2R$. Finally, a semisphere of radius R is mounted on the cylinder. Express the cross-sectional area S of the tower as a function of the distance x of the cross section from the lower base of the cone. Draw the graph of the function $S = f(x)$.

45. A cylinder is inscribed in a sphere of radius R. Express the volume V of the cylinder as a function of its height x. Indicate the domain of definition of this function.

46. A right cone is inscribed in a sphere of radius R. Find the functional relationship between the lateral surface area S of the cone and its generatrix x. Indicate the domain of definition of this function.

In Problems 47 and 48 find the domains of definition of the given functions.

47. (1) $y = 1 - \log_{10} x$; (2) $y = \log_{10}(x+3)$;
(3) $y = \sqrt{5-2x}$; (4) $y = \sqrt{-px}$ $(p > 0)$;
(5) $y = \frac{1}{x^2-1}$; (6) $y = \frac{1}{x^2+1}$; (7) $y = \frac{1}{x^3-x}$;
(8) $y = \frac{2x}{x^2-3x+2}$; (9) $y = 1 - \sqrt{1-x^2}$;
(10) $y = \frac{1}{\sqrt{x^2-4x}}$; (11) $y = \sqrt{x^2-4x+3}$;

(12) $y=\frac{x}{\sqrt{x^2-3x+2}}$; (13) $y=\arcsin\frac{x}{4}$;

(14) $y=\arcsin(x-2)$; (15) $y=\arccos(1-2x)$;

(16) $y=\arccos\frac{1-2x}{4}$; (17) $y=\arcsin\sqrt{2x}$;

(18) $y=\sqrt{1-|x|}$; (19) $y=\frac{1}{\sqrt{|x|-x}}$;

(20) $y=\frac{1}{\sqrt{x-|x|}}$; (21) $y=\sqrt{\log_{10}\left(\frac{5x-x^2}{4}\right)}$;

(22) $y=\log_{10}\sin x$; (23) $y=\arccos\frac{2}{2+\sin x}$;

(24) $y=\log_x 2$.

48. (1) $y=\frac{1}{\log_{10}(1-x)}+\sqrt{x+2}$;

(2) $y=\sqrt{3-x}+\arcsin\frac{3-2x}{5}$;

(3) $y=\arcsin\frac{x-3}{2}-\log_{10}(4-x)$;

(4) $y=\sqrt{x}+\sqrt[3]{\frac{1}{x-2}}-\log_{10}(2x-3)$;

(5) $y=\sqrt{x-1}+2\sqrt{1-x}+\sqrt{x^2+1}$;

(6) $y=\frac{3}{4-x^2}+\log_{10}(x^3-x)$;

(7) $y=\log_{10}\sin(x-3)+\sqrt{16-x^2}$

(8) $y=\sqrt{\sin x}+\sqrt{16-x^2}$;

(9) $y=\frac{1}{\sqrt{\sin x}}+\sqrt[3]{\sin x}$;

(10) $y=\log_{10}\frac{x-5}{x^2-10x+24}-\sqrt[3]{x+5}$

(11) $y=\sqrt{\frac{x-2}{x+2}}+\sqrt{\frac{1-x}{1+x}}$;

(12) $y=\sqrt{x^2-3x+2}+\frac{1}{\sqrt{3+2x-x^2}}$;

(13) $y=(x^2+x+1)^{-\frac{3}{2}}$;

(14) $y=\log_{10}(\sqrt{x-4}+\sqrt{6-x})$;

(15) $y=\log_{10}[1-\log_{10}(x^2-5x+16)]$.

49. Are the following functions identical?

(1) $f(x)=\frac{x}{x^2}$ and $\varphi(x)=\frac{1}{x}$; (2) $f(x)=\frac{x^2}{x}$ and $\varphi(x)=x$;

(3) $f(x)=x$ and $\varphi(x)=\sqrt{x^2}$; (4) $f(x)=\log_{10}x^2$ and $\varphi(x)=$ $=2\log_{10}x$?

50. Give an example of an analytically represented function:
(1) defined only on the interval $-2 \leqslant x \leqslant 2$;
(2) defined only on the interval $-2 < x < 2$ and not defined at $x = 0$;
(3) defined for all real values of x, except for $x = 2$, $x = 3$, $x = 4$.

51. Find the domains of definition of the single-valued branches of the function $y = \varphi(x)$ specified by the equation:
(1) $y^2 - 1 + \log_2(x - 1) = 0$; (2) $y^4 - 2xy^2 + x^2 - x = 0$.

Characteristics of behaviour of functions

52. $f(x) = \frac{x^2}{1+x^2}$; indicate the domain of definition of the function $f(x)$ and make sure that this function is nonnegative.

53. Find the intervals of constant sign and the roots of the functions:
(1) $y = 3x - 6$; (2) $y = x^2 - 5x + 6$; (3) $y = 2^{x-1}$;
(4) $y = x^3 - 3x^2 + 2x$; (5) $y = |x|$.

54. Which of the below functions are even, which are odd, and which are neither even nor odd?

(1) $y = x^4 - 2x^2$; (2) $y = x - x^2$; (3) $y = \cos x$;
(4) $y = 2^x$; (5) $y = x - \frac{x^3}{6} + \frac{x^5}{120}$; (6) $y = \sin x$;
(7) $y = \sin x - \cos x$; (8) $y = 1 - x^2$; (9) $y = \tan x$;
(10) $y = 2^{-x^2}$; (11) $y = \frac{a^x + a^{-x}}{2}$; (12) $y = \frac{a^x - a^{-x}}{2}$;
(13) $y = \frac{x}{a^x - 1}$; (14) $y = \frac{a^x + 1}{a^x - 1}$; (15) $y = x \cdot \frac{a^x - 1}{a^x + 1}$;
(16) $y = 2^{x - x^4}$; (17) $y = \ln \frac{1-x}{1+x}$.

55. Represent each of the following functions in the form of a sum of an even and an odd function:
(1) $y = x^2 + 3x + 2$; (2) $y = 1 - x^3 - x^4 - 2x^5$;
(3) $y = \sin 2x + \cos \frac{x}{2} + \tan x$.

56. Prove that $f(x) + f(-x)$ is an even function, and $f(x) - f(-x)$ is an odd one.

57. Represent the following functions as a sum of an even and an odd function:
(1) $y = a^x$; (2) $y = (1 + x)^{100}$ (see Problem 56).