Theoretical and Applied Mechanics

Proceedings of the 14 th IUTAM Congress
Delft, The Netherlands,
30 August–4 September 1976

W.T. KOITER, Editor

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GENERAL REPORT

After more than half a century the International Congress of Theoretical and Applied Mechanics returned to its native city Delft in the Netherlands. Earlier invitations by the Division for Mechanics of the Royal Institution of Engineers in the Netherlands for the 12-th and 13-th Congresses were withdrawn in favour of Stanford University in 1968 and Moscow University in 1972.

As in previous Congresses the programme was established in a close consultation between the Local Organizing Committee and the Congress Committee of IUTAM. The Executive Committee of these two bodies had five joint meetings. It was agreed that the backbone of the Congress would consist of five general lectures and thirty sectional lectures, each of one hour's duration. The general lectures and about half of the sectional lectures were delivered by invitation by the Congress Committee of IUTAM. The second half of the sectional lectures were selected by an International Papers Committee from about 255 contributed papers accepted for presentation at the Congress. Nearly 800 summaries were submitted in response to the call for papers, and the International Papers Committee regretted the necessity to reject many good proposals. It was agreed, however, that half an hour should be allocated to each accepted paper, and the programme would have been overloaded by the acceptance of a larger number of contributed papers.

It was also agreed that more time should be made available for discussions than in previous Congresses. Prepared discussions were arranged for the sectional lectures, introduced by the chairmen of the sessions in question, and a period for open discussions of the papers in each of the seven parallel sessions was scheduled at the end of each Congressday.

Restrictions on travel grants in many countries resulted in a smaller number of participants than anticipated originally. It is not felt that this smaller number of Congress members, slightly over one thousand, including the Congress staff members recruited from Delft University of Technology who were registered free of charge, made the Congress less effective. The Organizing Committee gratefully acknowledges its indebtedness to Delft University of Technology, the Netherlands Government and Industry for their material assistance in overcoming the budgetary problems resulting from the smaller number of participants. Thanks are also due to IUTAM and the ICSU Committee on Science and Technology in Developing Countries for their financial assistance to young scientists, in particular from developing countries, enabling 45 younger scientists to attend the Congress.

In the opening session Congress members were welcomed on behalf of the Netherlands Government by Dr. G. Klein, State Secretary for Science and Education. The working sessions of the Congress were held in the magnificent Main Auditorium building of Delft University of Technology and in the adjacent building of the Department of Mechanical Engineering. The facilities included a closed TV circuit for the information of Congress members of programme changes etc., pigeon holes for the mutual communication between participants, and a xerox installation for the reproduction of papers. The effective assistance by Auditorium Staff members and by the Congress Offices of the Ministry of Education and Sciences and of the Royal Institution of Engineers in the Netherlands ensured a smooth running of the entire registration and information process.

The social programme included a reception by the Netherlands Government in the venerable Knights'Hall in The Hague, an excursion to Amsterdam, and a 'Dutch Evening' in the Congress Centre in The Hague. All these events and the additional Ladies'Programme were enjoyed by large numbers of participants.

The editor is indebted to all committee members who took part in the establishment of the programme and in the organisation of the Congress. Special thanks are due to Professor Ernst Becker, secretary of the IUTAM Congress Committee, and to Dr. Arnold M.A. van der Heijden, secretary of the Local Organizing Committee. Finally, the editor wishes to express his gratitude to North-Holland Publishing Company for their most helpful service and effective production of the Proceedings.

W.T. Koiter.

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Delft University of Technology has provided its outstanding facilities for the Congress. The Ministry of Education and Science and the Royal Institution of Engineers in the Netherlands (KIvI) have put their Congress Offices at the disposal of the Congress. The Ministry of Education and Science, Delft University of Technology and industry have rendered financial support or guarantees to the Congress. The industrial companies involved are listed in alphabetical order:

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The agency "Convention Travel International B.V." has been in charge of hotel accommodation etc. The official carrier of the Congress is KLM Royal Dutch Airlines.

The Netherlands Government have graciously offered their hospitality in the reception in the Knights' Hall ("Ridderzaal") in The Hague on Wednesday, 1 September.

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MATHEMATICS, MECHANICS AND OUR CONCEPTION OF THE PHYSICAL WORLD

H.B.G. Casimir

Heeze, The Netherlands

When, in 1958, during the celebration of the 150th anniversary of our Royal Academy of Science I had the privilege to speak on "Certainty in the Exact Sciences", I took the examples by means of which I tried to elucidate the nature of physical theory mainly from mechanics. And for good reason. Mechanics is not only the first example of a quantitative mathematical theory describing physical phenomena, it has also been a most fruitful one and it has had a profound influence on our way of thinking about nature; ever since Newton published his "Philosophiae Naturalis Principia Mathematica" mechanics has been an essential component of natural philosophy.

As I explained in my 1958 lecture our confidence in physical theory is based on the possibility of reproducible experiments with quantitative results, identical for all observers, the possibility of deducing such predictions from logical, mathematically formulated theories. Reproducibility, prediction and mathematical formulation are the hall-mark of the disciplines of physics; mechanics was the first discipline to meet these standards.

The simplifications that make physics possible are not permissible when we are dealing with living beings. No two pebbles are identical, but they can be sufficiently identical to behave in exactly the same way when they are falling. No two men are identical and their idiosyncrasy may well be of decisive influence on the course of a disease or the success of a surgical operation. A physical system can be separated from its surroundings. A human being in complete isolation finds itself in a very abnormal condition. Therefore one should be very cautious when applying the methods of mechanics to the study of living beings. In the course of this talk I shall try to indicate that even the description of physical phenomena along the lines of mechanics has its limitations.

The fundamental ideas of Newtonian mechanics could not only be applied to the motion of all celestial and terrestrial solid bodies, but also to the internal deformation of solids (theory of elasticity) and the internal motions in liquids and gases (hydrodynamics and aerodynamics). More and more one was inclined to look on mechanics not only as the number one physical theory but as the only physical theory.

Newton thought of light in terms of particles. Huygens considered wavefronts, but certainly not mathematical abstractions: they had to be waves in some medium. It is an amusing twist of history that Hamilton established complete mathematical identity between the dynamics of particles and geometrical optics just after Young and Fresnel had given convincing proof of the wave nature of light.

During the nineteenth century we see two important new developments. Thermal phenomena are explained in terms of motion of molecules; Tyndall's "Heat, a mode of motion" epitomizes the main idea. Electromagnetic phenomena are explained in terms of an aether, a curious medium with characteristics different from any substance but still a medium. Light is explained in terms of electromagnetic waves. To put it slightly differently: we witness the extension of particle dynamics to the motion of hypothetical atoms and molecules, but also the description of the electromagnetic field by methods akin to - though not entirely identical with - the dynamics of continuous media. Throughout this development the principles of reproducibility, prediction and mathematical formulation, first encountered in mechanics, prevail. Yet a new element appears: the notion of statistical fluctuations when we are dealing with large numbers of particles.

Let us now look at modern physics. From 1900 onwards we have witnessed a curious development. The hypothetical particles, molecules, atoms, electrons became more and more "real". Their number, size and mass could accurately be determined and in many cases single particles could be counted. But at the same time it became increasingly clear that their behaviour could not be described by mechanics as we knew it. This resulted finally in an entirely new mathematical discipline, quantummechanics, of which classical mechanics is a limiting case. Quantum mechanics owes much to the mathematical formalism of classical mechanics, but it has also led to a way of interpreting the formalism that is entirely different from classical mechanics.

As to the theory of the electromagnetic field, there the idea of a substantial aether was abolished by Einstein's relativity theory, which also brounght a modification of the equations of mechanics, be it a far less radical one than quantummechanics. But even apart from relativity the idea of an aether became gradually bizarre: one had learnt to understand the properties of solids and fluids in terms of atomic particles with electromagnetic interactions, therefore it was illogical to want to reduce these interactions to the properties of a medium.

So we are left with particles obeying entirely new laws of motion and the electromagnetic field, a mathematical abstraction that provides the interaction between the particles. Or, rather, I should say fields, for nuclear physics has revealed there are other interactions besides the electromagnetic one.

Quantummechanics is not only a different formalism, it has also led to a new interpretation, a new way of looking at nature. The following rather sketchy description will not do too much injustice to theory. Particles and their movement are described by "wave functions". These functions obey partial differential equations and are therefore analogous to, for instance, the pressure in a continuous medium. But these wavefunctions themselves do not correspond to any measurable quantity. The square of the wavefunction - or rather the modulus of the square, for the formalism is such that the wavefunctions are complex quantities - determines the probability that a particle is there. Here the notion of probability enters in an entirely new way: not as an average because we are dealing with a large number of factors that we do not know, but that might be known in principle, but as an essential element of our description of nature. As a consequence strict causality disappears. The future cannot be predicted with certainty from initial conditions and this uncertainty is not the result of the complicated structure of a system of many components, it is already there for one single particle. This is a radical departure from the ideas of classical mechanics.

Not all physicists accepted this interpretation of quantum mechanics as a final answer. Einstein did not accept it and his discussions with Bohr, who did, are a most fascinating chapter in the history of physics. After looking in vain for inconsistencies in the statistical interpretation of quantum mechanics - every objection he could think of was invalidated by Bohr - Einstein argued that a theory like this may well describe a large number of phenomena, but that it cannot be the definitive theory. According to Einstein we should try to know even more about atomic particles and then we should be able to make exact predictions, more or less along the lines of classical mechanics. Bohr believed that the limitations of physical theory are essential and permanent.

So far Bohr's point of view has been by far more fertile and most of today's physicists would subscribe to Bohr's point of view. But I do not want to make predictions about the future: the interpretation may well change. However, I feel certain that future generations of physicists will continue to use successfully the mathematical formalism of quantum mechanics, just as we are successfully continuing to use Newtonian mechanics, although we know it neglects quantum effects.

I have tried to sketch the influence of mechanics on our way of thinking about nature. This congress will bear witness to the fruitfulness of its application to a wide range of problems.

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J.H. ARGYRIS

Computer and Mechanics.

The paper gives an account, with numerical examples, of recent experiences at ISD in the treatment of non-linear and some special linear problems. The topics covered are as follows:

a) Large deflection and post-buckling analysis of structures.

Various problems of beams, arches and shells are solved using the natural mode method. Noteworthy points are the unsymmetrical geometrical stiffnesses that may occur in three-dimensions when using rotational nodal freedoms, and the introduction of a very simple flat facet shell element of engineering precision. The latter element is partly based on physical lumping and some observations on this much neglected method are included in the paper. Post-buckling is taken to deflections of the order of the structure dimensions.

b) Incompressible elastic material problems.

Finite element treatment of incompressible material may impose complete incompressibility, or a variational constraint on compressibility, or may use a material with elastic constants imposing near incompressibility. Each procedure has its physical and numerical "pros and contras". In the present paper an algorithm for the efficient handling of systems with a large number of rigid constraints is applied to the problem of absolutely incompressible elastic material. Advantage of method is that the elimination of the constraints gives a system without the high dilational frequencies of the nearly incompressible material. This eases the dynamic response problem by permitting larger time steps in the integrating algorithm.

c) Plane incompressible viscous flow.

This problem has become somewhat of a test case for the various numerical methods that have been proposed for its solution. It has been suggested by more than one author that many conventional finite element (and finite difference) schemes may produce garbage at high Reynold's Number. In the paper some previous work, in which global polynomial functions of the stream function only were applied to a plate analogy, is extended to numerical experiments using global Tchebychev polynomials and cubic finite elements in an infinite channel and TUBA 6 finite elements in a finite channel.

d) Large strain elasto-plasticity.

In recent years there has been much interest in theories of metal plasticity with both elastic and inelastic finite strain components; in particular the literature contains several important dynamic applications such as high-speed impacts, explosive metal forming and armour penetration. These phenomena falling outside of the classical theories have led investigators to seek a more general theory of elastic-plastic media with finite strains which reduces to the classical theory under infinitesimal strains. The paper presents a so-called "natural" discretized formulation describing idealized elastic-plastic materials subjected to arbitrary large strains. This formulation has already proven its numerical efficiency and theoretical clarity when applied to elastic problems. The commonly accepted additivity of the Green (material) or Almansi (spatial) strain rate measures is not assumed, although additivity of certain other strain measures is postulated. Different hardening models are taken into account. We describe the algorithm leading to the solution of such a problem and compare it with the algorithms which are commonly used in solving elasto-plastic problems. Two examples are presented: one resembling a cable roof structure and the other exploring the necking process of a strip under tension.

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BIOMECHANICS

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1. INTRODUCTION

Biomechanics seeks to understand the mechanics of the living. It is an ancient subject, and covers a very wide territory, which ranges from subcellular components to single cells to plants and animals. In the last IUTAM Congress in 1972, Lighthill [1] spoke about the locomotion of aquatic animals, Goldsmith [2] spoke about the flow of erythrocytes. Today I shall report on physiological and medical applications; which constitute the majority of recent work in this field. The impetus for research in this area comes from the realization that physiology cannot be understood without biomechanics, not any more than an airplane can be understood without aerodynamics. For an airplane, mechanics enables us to design its structure and predict its performance. For an organ, biomechanics may help us understand its normal function, predict changes due to alterations, and propose methods of artificial intervention. Thus, diagnosis, surgery and prosthesis are closely associated with biomechanics.

The applied mechanics community is generally familiar with the contributions of Galileo (1564-1642) to the measurement of heart rate, of Descartes (1596-1650) to the analysis of the eye, of Borelli (1608-1679) to the analysis of the limbs, of Robert Hooke (1635-1703) to the observation on cells, of Euler (1707-1783) to the analysis of pulse waves in the arteries, of Thomas Young (1773-1829) to the theory of voice and vision, and of Helmholtz (1821-1894) to the theories of speech, vision, and psychophysiology. You will probably be happy to learn that the high frequency waves in the arteries predicted by Lamb (1849-1934) were found in recent experiments. You will be equally pleased to learn that many famous physiologists had their reputation established on mechanics. Thus Stephen Hales (1677-1761) measured the arterial blood pressure and correlated it to haemorrhage, measured the distensibility of the heart and aorta and used it to explain the "smoothing" action of the aorta in converting the pulsatile flow of the heart to a smooth flow in blood vessels. He introduced the concept of peripheral resistance in blood flow, and showed that the main site of this resistance was in the minute vessels in the tissue. Poiseuille (1799-1869(clarified the concept of viscosity and resistance in blood flow. Otto Frank (1865-1944) clarified the mechanics of the heart. Starling (1866-1926) proposed the law for mass transfer across a membrane and clarified the water balance in our body. Krogh (1874-1949) won his Nobel prize on the mechanics of microcirculation. Hill (1886-) won his Nobel prize on the mechanics of the muscle.

Traditionally, engineering and medicine developed independently that a language barrier arose. Thus a special application of the law of conservation of mass is called Fick's principle in cardiology; a common formula to compute the membrane stress in a spherical shell is called the law of Laplace. Such redundancy adds confusion to our subject. On the other hand, each organ has its particular structure and function, each tissue has its own properties. Without a general knowledge about the specific organ we are dealing with, it is difficult to make definitive contributions. Thus biomechanics needs anatomy, histology, biochemistry, physiology, and pathology. We need biological details to define boundary value problems. It turns out that some of the details required for the formulation of biomechanical problems are often not available from the literature. Thus it is almost a universal experience of biomechanics workers to have to do some studies normally associated with other disciplines.

Against such a general background, I would like to describe some recent developments in this field with emphasis on the characteristics of the mechanical properties of living tissues, and the nature of the boundary value problems that arise. In Sec. 2, we shall discuss the constitutive equations of soft tissues. In Sec. 3, we present a typical problem concerning a vital organ, the lung, in order to demonstrate how theoretical and applied mechanics can be enriched by biology. The revers question, what has biomechanics contributed to health science, will be answered in the last section.

2. STRESS-STRAIN RELATIONSHIP FOR LIVING SOFT TISSUES

Mechanical changes in tissues and organs are important to medicine because the manifestations of many diseases consist of changes in mechanical properties. For example, arteriosclerosis is a thickening and change of the artery. Hyaline membrane disease is a condition of the infant's lung in which the alveolar wall looks glassy. Fibrosis is growth of white fibrous connective tissue in an organ. These changes in tissues cause changes in their rheological properties. Functional changes of the organs follow rheological changes in their tissues.

A mathematical constitutive equation is needed for the analysis of the mechanics of the organs. Since it is intended to be determined in the laboratory or on the patient, and used in further mathematical analysis, it is obvious that their form should be simple. The empirical constants should be as few as possible. This practical consideration guides our choice of approximate constitutive equations.

2a. Empirical Information

Biological tissues have complex structures and are usually capable of large deformation with finite strain. Their stress-strain relationships are in general non-linear. For example, Fig. 1 shows the length-tension diagram of a circumferential strip of dog's aortic arch [3]. The tension can be converted to stress according to Lagrange's definition, by dividing the tension with the cross sectional area of the aortic strip in the resting condition (unstrained). The length changes can be converted to the stretch ratio λ by dividing the stretched length with the resting length of the specimen. The resting length, as well as the data in Fig. 1, were obtained after a number of repeated loading and unloading cycles. If we compute the slope of the loading branch of the stress-strain curve and plot it against the tension in the specimen, we obtain the results as shown in Fig. 2. It is seen that for stresses (defined in Lagrangian sense) higher than 200 g/cm^2, the tangent modulus varies almost linearly with the tensile stress. But for smaller stresses the tangent modulus decreased rapidly; i.e., the tissue becomes very soft at very low stresses.

Thus, arteries show nonlinear stress-strain relationship. Other soft tissues show an even stronger nonlinearity. Figure 3 shows the stress-strain curves of a papillary muscle of the rabbit heart in tension [4]. It shows loading-unloading at three different strain rates: each of which was recorded after a preconditioning process. If the tangent modulus of the papillary muscle on loading is plotted against the tension as in Fig. 4, we see an almost straight line which does not, however, pass through the origin [4]. Other tissues, such as the lung [5, 6], the ureter [7] and the mesentery [8], behave like the papillary muscle. The myosin fibers [9, 8] and even the tendon at a lower stress level (Stromberg et al. [10]), behave similarly. For these tissues [8], the tangent-modulus-vs-tension curves for loading after preconditioning may be represented by the following equation in limited range of stress and strain:

$$\frac{dT}{d\lambda} = \alpha(T+\beta)\ , \qquad (0 < T_a \le T \le T_b) \qquad (1)$$

i.e.

$$T = (T^* + \beta)e^{\alpha(\lambda-\lambda^*)} - \beta,\ (1 < \lambda_a \le \lambda \le \lambda_b) \qquad (2)$$

where λ^*, λ_a, λ_b corresponds with T^*, T_a and T_b respectively.

The exponential type of stress-strain relationship can be generalized to two- and three-dimensions. For example, the extensive experimental results of Lanir and Fung [11] on the skin of rabbit's abdomen can be fitted by the following stress-strain relationship [12]:

$$\sigma_1 = \frac{\partial W}{\partial e_1}\ , \qquad \sigma_2 = \frac{\partial W}{\partial e_2}, \qquad (3)$$

where e_1, e_2 are the finite strain components corresponding to the stretch ratios λ_1, λ_2, respectively:

$$e_1 = \frac{1}{2}(\lambda_1^2 - 1), \qquad e_2 = \frac{1}{2}(\lambda_2^2 - 1), \qquad (4)$$

and W is the two-dimensional strain-energy function:

$$W = \frac{1}{2}(\alpha_1 e_1^2 + 2\alpha_4 e_1 e_2 + \alpha_1 e_2^2) + \frac{1}{2} C \exp[a_1 e_1^2 + a_2 e_2^2 + 2a_4 e_1 e_2 + e_1 e_2(\beta e_1 + \beta_2 e_2)] \qquad (5)$$

The constants α_1, α_4, C, a_1, a_4, β_1, β_2 must be determined experimentally. A typical set of values for the skin is

$$\alpha_1 = 13.8 \qquad \alpha_4 = 7.8 \qquad (6)$$

$$a_1 = 3.904, \quad a_2 = 11.28, \quad a_4 = -13.97$$

$$\beta_1 = 24.85, \quad \beta_2 = 32.77, \quad C = 0.0164$$