Eberhard Zeidler

Nonlinear Functional Analysis and its Applications

I: Fixed-Point Theorems

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Translated by Peter R. Wadsack

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Preface

What is clear and easy to grasp attracts us; complications deter.
David Hilbert

Everything should be made as simple as possible, but not simpler.
Albert Einstein

Science will not light the lamp in a person whose soul has no fuel.
Michel de Montaigne

In the course of the last 15 years I have had the opportunity to lecture on a variety of topics in nonlinear functional analysis and its applications. In each case, I was able to recommend to my students a series of outstanding monographs on the particular problems under consideration. What was missing was a comprehensive treatment of nonlinear functional analysis, accessible to a broader audience of mathematicians, natural scientists, and engineers, with a command of the basics of linear functional analysis only, which would provide a rapid survey of the subject. I attempted to close this gap with a five-part expansion of my lecture notes. The first three parts appeared as Teubner texts (Teubner-Verlag, Leipzig, 1976, 1977, 1978). The present English edition was translated from a completely newly written manuscript which represents a significant enlargement and revision of the original version. The material is organized as follows:

Part I: Fixed-Point Theorems;
Part II: Monotone Operators;
Part III: Variational Methods and Optimization;
Parts IV/V: Applications in Mathematical Physics.

A Table of Contents of Parts II–V can be found on p. 871.

All of the necessary basic tools from linear functional analysis are contained in the appendix to this volume, where they are summarized along with typical examples. Thus the basic content of all five volumes can be understood even by those readers who have little or no knowledge of linear functional analysis. Such a reader will find detailed instructions in the introduction to the Appendix to this volume.

The emphasis of the treatment is based on the following considerations:

Which are the basic, guiding concepts, and what relationship exists between them?

What is the relationship between these ideas and the known results of classical analysis and of linear functional analysis?

What are some typical applications?

Through all of this, the reader is intended to feel that the theory is being developed, not simply for its own sake, but with an eye toward finding effective solutions of concrete problems.

We will attempt to illuminate the subject from many sides—from the set-theoretic foundations (the Bourbaki–Kneser Fixed-Point Theorem) all the way to concrete numerical methods, and their numerous applications in physics, chemistry, biology, and economics. The reader should then begin to see the mathematics involved as a unified whole. At the same time, we want to show how deep mathematical methods can be applied in the natural sciences, in technology, and in economics. The development of nonlinear functional analysis has been substantially influenced by complex problems posed by the natural sciences, and in its continued development, a close contact with the natural sciences will be of great significance. In the presentation we have chosen here, the emphasis is on the use of analytical methods, although we will also attempt to show the relationships with algebraic and differential topology. Furthermore, the presentation has been influenced strongly by the spirit of modern global analysis.

We make no attempt to deal in the broadest generalities, but rather, we will try to expose the essential core simply, without trivializing it. In the experience of the author, it is substantially easier for the student to take a mathematical concept and extend it to a more general situation, than to struggle through a theorem formulated in its broadest generality and burdened with numerous technicalities in an attempt to divine the basic concept. Here it is the teacher's duty to be helpful. To assist the reader in recognizing the central results, these propositions are denoted as theorems. In an appendix to the Table of Contents, there is an index of these theorems and of the basic definitions, beginning on p. 859. In establishing such a list, there is, needless to say, an element of discretion.

Every chapter is self-contained. Each begins with motivations, heuristic approaches, and indications of the typical problems to be solved. Each contains the correspondingly most important propositions and definitions, together with clarifying examples, illustrations, and typical applications. In the interests of the reader, we have no qualms about including quite simple examples. We also try to get to the heart of the matter as rapidly as possible. Finally, we think it important that the reader know, at each stage of the book, how the general theory applies to specific concrete applications. All of this required a very careful selection of material, since one could easily write a specialized monograph for each chapter. Indeed, such monographs already exist in part. At the end of each chapter, there are problems and a bibliography. The problems vary substantially in difficulty:

Problems without a star are for practice in using the material presented, and require no additional methods.

Problems with one star are more difficult. The solution requires additional ideas.

Problems with two stars are very difficult, and their solution requires extensive additional knowledge.

Each problem contains either the solution or an exact reference to the monograph or original paper in which the solution may be found. Additionally, we provide commentary designed to clarify the significance of the results. Problems with one or two stars could also be regarded as references aimed at important extensions of the results.

The references at the end of each chapter are of the following form: Krasnoselskii (1956, M, B, H); etc. The name and the year relate to the bibliography at the end of the book. The letters stand for the following:

M: monograph;
L: lecture notes;
S: survey article;
P: proceedings;
B: extensive bibliography in the work cited;
H: comments on the historical development of the subject contained in the work cited.

Finally, we describe the common thread of the works cited. Given the expanse of the existing literature, it was necessary to make a careful selection, and these choices are those which, in the necessarily subjective opinion of the author, provide the reader with the easiest access to a comprehensive picture of continuing results. There is a natural emphasis, therefore, on survey articles and monographs. However, we also cite a number of classical works which were of special significance for the development of nonlinear functional analysis. We recommend that the reader look at at least a few of these works, so as to get a lively impression of the creative process and the historic development of the mathematics that lead to these new results. In order to keep the bibliography within reasonable bounds, many important works had to be omitted. Some chapters contain various general references for, for example, the theory of integral equations, of ordinary and partial differential equations, of numerical methods, of algebraic or differential topology, etc. On p. 865 there is a listing of these general references.

In order to keep the book as self-contained as possible, we have included a number of results from linear functional analysis in the appendices to Parts I–III.

We recommend that the reader begin directly with the text of a chapter, and only refer to the appendix upon discovering a gap in knowlege. A reference of the form A_i (20) is to (20) in the appendix, Part i, where $i = 1, \ldots, 5$, while (16.20) refers to formula (20) in Chapter 16. Omission of a chapter number means that the formula is in the current chapter. In each chapter, the main results (i.e., the theorems) are distinguished by upper-case roman letters; for instance, "Theorem 12.B in 12.7" means the second theorem in Chapter 12, located in Section 7 of that chapter. However, propositions, definitions, examples, etc., are numbered consecutively in each chapter; for example, in Chapter 14 one finds Example 14.1, Proposition 14.2, Example 14.3, Definition 14.4, etc., in that order. A reference to Proposition 2.6(I) is to step (I) in the proof of Proposition 2.6. We subdivide the chapters among the five parts of this work in the following way.

Part I: Chapters 1–17;
Part II: Chapters 18–36;
Part III: Chapters 37–57;
Part IV: Chapters 58–79;
Part V: Chapters 80–100.

An index of symbols used may be found on p. 851. We have tried to use generally accepted symbols. A few peculiarities, introduced to avoid confusion, are described in the remarks introducing the symbol index. As for abbreviations, we basically use B-space and H-space for Banach and Hilbert spaces, respectively, and M–S sequence for Moore–Smith sequence.

We developed our presentation with due consideration to the fact that a book is rarely read straight through from beginning to end. We hope that even a cursory skimming of the text will suffice to impart the basic content. For such an approach, we recommend reading the chapter introductions, the definitions, the theorems and propositions (but not the proofs), the examples (but not the proofs), as well as the numerous remarks interspersed throughout the text, which deal with the significance of the individual results. The reader who does not have time to solve the exercises should nevertheless skim the headings for the problems, as well as the remarks, which describe the significance of the problem and its relationship to the material. The reader who is interested in supplementary problem material can try to prove independently all of the numerous examples in the text, without first peeking at the proofs provided. All hypotheses of a proposition are explicitly stated, so that there is no need for a time-consuming search of the antecedent text to find them. We have attempted to reduce the number of definitions to a minimum, so as not to overburden the reader with too many concepts. In order to clarify connections, related results are at times gathered into a single proposition. This approach was chosen in part for the benefit of natural scientists and engineers, whose primary interest is to find out what help mathematics can provide for various nonlinear problems. For a very quick reading, it sufficies to look at the theorems and the corresponding definitions.

The proofs are highly structured, to help the reader who is interested in proofs in recognizing the individual steps and ideas involved. As is well known, a careful study of proofs is the only road to a deeper understanding of mathematical theory.

The contents of the individual chapters can be gleaned from the extensive Table of Contents on p. 871. A brief summary of the general goals follows.

Part I consists of three sections:

fundamental fixed-point principles;
applications of the fundamental fixed-point principles;
mapping degree and fixed-point index.

We begin with the three fundamental fixed-point results: the fixed theorems of Banach, Schauder, and Bourbaki–Kneser. From these we derive a number of important results without using the mapping degree. This approach was chosen for didactic reasons. The reader who wishes to learn about the mapping degree immediately may begin reading in Chapter 12. The applications in Part I are concentrated primarily on differential and integral equations in Banach spaces of sufficiently smooth functions, without using Lebesgue integrals or Sobolev spaces; applications concerning the latter are included in Parts II–V. We place special emphasis on stability questions. The applications in the sciences are in nonlinear oscillation, heat conduction, ecological and economic models, game theory, chemical reactions, minimal surfaces, boundary layer equations, representation theory of Lie groups and the classification of elementary particles and molecular vibrations, problems in celestial mechanics, interval mathematics, formal computer languages, foundations of set theory, etc. A number of deeper applications could only be indicated in the problem sections, due to a lack of space. Further applications belonging to this problem area can be found in Parts IV and V.

Part II, which deals with monotone operators, consists of sections on

an introduction to the subject;
an examination of linear problems;
a generalization to nonlinear stationary problems;
a generalization to nonlinear nonstationary problems; and
a general theory of discretization methods.

The theory of monotone operators, as developed in the last 20 years, represents a natural generalization from Hilbert space methods for linear differential and integral equations to nonlinear problems, and is therefore of prime significance for the solution of numerous applied nonlinear problems. Our main goal in Part II is to clarify the connection between classical linear Hilbert space methods and the theory of monotone operators. The significance of these methods arises from a background formed by the physical concept of energy. In Part II we consider applications to differential and integral equations, using Sobolev and Lebesgue spaces. Familiarity with these spaces is not a prerequisite, since the necessary material is contained in the text.

Part III, on variational methods and optimization, consists of the following sections:

an introduction to the subject;
extremal problems without side conditions;
extremal problems with smooth side conditions;
extremal problems with general side conditions;
saddle points and duality; and
variational inequalities.

Here it is our goal to develop the common ground between classical variational methods and modern optimization methods, with the aid of nonlinear functional analysis. There are numerous applications to differential and integral equations, differential inequalities, optimization problems, control problems, variational problems, approximation theory, information theory, statistical physics, game theory, etc.

Parts IV and V are dedicated to the deeper applications of nonlinear functional analysis to mathematical physics. Here we place particular emphasis on the derivation of the basic equations of, for example, mechanics, nonlinear elasticity and plasticity theory, hydro- and gas dynamics, thermodynamics, statistical physics, kinetics of chemical reactions, general relativity, electrodynamics, quantum theory, etc. We hope to provide the reader with an understanding, not only of the mathematical problems, but also of the physical interpretation of the mathematical results. At the end of Part V there is a brief outline of the history of nonlinear functional analysis, which is intended to clarify the historical lines of development.

Besides the natural science applications, we steadily emphasize methods for constructing approximate solutions, along with convergence proofs. In this context, our main concern is to develop the central principles of approximation methods within the framework of numerical functional analysis, such as the stability of fixed points and iterative methods, or the connection between convergence, consistency, stability, and existence in projection methods. We will also emphasize the use of tools from differential topology in modern numerical mathematics. As every numerical analyst well knows, the difficulties are in the particulars, so that, in principle, each problem requires its own specific numerical approach. The interested reader can pursue this in F. S. Acton's book, *Numerical Mathematics that Work* (1970), or in G. E. Forsythe's survey acticle, *Pitfalls in computation, or why a math book isn't enough* (1970). Nevertheless, an understanding of the general principles is helpful in organizing and relating the numerous existing concrete numerical methods within a general framework.

For the convenience of the reader, I have tried to keep Parts I–V, as well as the individual chapters, as independent of each other as possible.

The study of a mathematical text always demands hard work. It is my hope that the reader, after doing that work, will be delighted by the discovery of new insight into a modern, multifaceted mathematical discipline and its applications. I have tried not just to organize the propositions and definitions, but to impart to the reader much of the general experience accumulated over many years by the many mathematicians who have devoted themselves to nonlinear functional analysis. In doing this, I have also tried to distill clearly the general principles and strategies involved. The field is in a stormy state of development. This treatise contains the products of a long distillation—those which serve as a foundation for the entire theory. I hope that the reader, after perusing these lectures, will be in a position to pursue further developments independently to assign new results to their place in the existing theory, and to recognize genuinely new concepts as such.

Any critiques, suggestions, or remarks will be gratefully received.

I am deeply obligated to numerous colleagues, here and abroad, for interesting conversations and letters, as well as the papers and books they sent me. My special thanks is due my teacher, Prof. Herbert Beckert, for all which I was able to learn from him. In Leipzig, he trained a generation of mathematicians to concentrate on the essential mathematical substance and to avoid the pitfalls of overspecialization. I hope that some of that spirit appears in these volumes. Extensive thanks is also due Paul H. Rabinowitz and the Department of Mathematics at the University of Wisconsin–Madison (U.S.A.) for the invitation to a 4-month-long stay in the fall semester of 1978. This visit had substantial influence on the final form of this book. In the technical preparation of the manuscript I had the assistance of numerous colleagues. For typing parts of the manuscript, my grateful thanks to Ursula Abraham, Amira Costa, Elvira Krakowitzki–Herfurth, Heidi Kühn, Hiltraud Lehmann, Karin Quasthoff, Karla Rietz, Stefan Ackerman, Werner Berndt, Peter Fuhrmann, Thomas Hesse, Thomas Herfurth, Jürgen Köllner, Jürgen Schmidt, Rainer Schumann. For making photocopies, my grateful thanks to Sonja Bruchholz, and for calculating numerical examples on the computer, my grateful thanks to Inge Girlich, Johannes Maul, and Herbert Ristock. The understanding and experienced support of our department's librarian, Frau Ina Letzel, was very valuable to me. Furthermore, I would like to thank the administrators of our Department of Mathematics and its Director, Prof. Horst Schumann, for supporting the project.

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Eberhard Zeidler
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